Revisit:
Optimal interpolation in mapping of high-frequency radar-derived surface radial velocity maps

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Outline

• Formulation of least-squares fit (LSF) and optimal interpolation (OI) via inverse method.
• Comparison of correlation functions, vector current maps, uncertainty ellipses
• Summary
Radial velocity maps

- Radial velocity maps
- Range
- Bearing angle
- Scalar velocity

true vector

R1
R2
R3
Radial velocity maps

- Range
- Bearing angle
- Scalar velocity

true vector
Vector current maps

- Least-squares fit
  - Best fit of \( u \) and \( v \) with available radials within a search radius
- Spatially and temporally noisy data due to outliers
- Baseline inconsistency
- Divergence and vorticity should be estimated from vector current maps
Formulation (LSF vs. OI)

**Least-squares-fit (LSF)**

\[
\mathbf{d} = \mathbf{G}_a \mathbf{m}_a + \mathbf{e}
\]

\[
\begin{align*}
\mathbf{d}_1 &= u_1 \cos \theta_1 + v_1 \sin \theta_1 + e_1 \\
\mathbf{d}_2 &= u_2 \cos \theta_2 + v_2 \sin \theta_2 + e_2 \\
\mathbf{d}_3 &= u_3 \cos \theta_3 + v_3 \sin \theta_3 + e_3
\end{align*}
\]

\[
\hat{\mathbf{m}} = (\mathbf{G}_a^T \mathbf{G}_a)^{-1} \mathbf{G}_a^T \mathbf{d}
\]

**Optimal interpolation (OI)**

\[
\mathbf{d} = \mathbf{G} \mathbf{m}
\]

\[
\begin{align*}
\mathbf{d}_1 &= u_1 \cos \theta_1 + v_1 \sin \theta_1 \\
\mathbf{d}_2 &= u_2 \cos \theta_2 + v_2 \sin \theta_2 \\
\mathbf{d}_3 &= u_3 \cos \theta_3 + v_3 \sin \theta_3
\end{align*}
\]

\[
\hat{\mathbf{m}} = \mathbf{P} \mathbf{G}^T (\mathbf{G} \mathbf{P} \mathbf{G}^T + \mathbf{R})^{-1} \mathbf{d}
\]

\[
= (\mathbf{G}^T \mathbf{R}^{-1} \mathbf{G} + \mathbf{P}^{-1})^{-1} \mathbf{G}^T \mathbf{R}^{-1} \mathbf{d}
\]

P = *a priori* model covariance

R = observational error covariance

(Kim et al, JGR 2008)
Correlation functions (LSF vs. OI)

- Constant velocity within a search range: segmented correlation function
- An example of 1.5 km search radius

- Spatial weighting by distance: exponential, Gaussian, or observed covariance function
- An example of an exponential corr. function with 2 km decorrelation length scale

Examples of 25 MHz system with 1 km vector grid
Radial velocity maps
Vector current maps (LSF vs. OI)

- Spatially smooth vector current field
- Improved baseline inconsistency
- A search radius in OI is adopted to reduce the calculation time.
Uncertainty (LSF vs. OI)

Uncertainty estimate *(a posteriori error covariance)*

\[ \hat{P} = (G_a^T G_a)^{-1} \quad \hat{P} = (G^T R^{-1} G + P^{-1})^{-1} \]

Uncertainty normalized with error variance

\[ \varphi = (G_a^T G_a)^{-1} \quad \varphi = (G^T G + P^{-1} \sigma_r^2 I)^{-1} \]

Uncertainty normalized with model variance

? \[ \hat{\chi} = P^{-1/2} \hat{P} P^{-1/2} \]

\[ P = \text{a priori model covariance} = \sigma_s^2 I \]
\[ R = \text{observational error covariance} = \sigma_r^2 I \]
Uncertainty (LSF vs. OI)

\[ \nu = (G^\dagger G)^{-1} = \begin{bmatrix} \nu_{uu} & \nu_{uv} \\ \nu_{vu} & \nu_{vv} \end{bmatrix} \]

\[ \kappa = \begin{bmatrix} \kappa_{uu} & \kappa_{uv} \\ \kappa_{vu} & \kappa_{vv} \end{bmatrix} \]

\[ \nu_{uu} = \frac{1}{\text{det}(G^\dagger G)} \sum_{l=1}^{L_a} \sin^2 \theta_l \geq \frac{1}{L_a} \]

\[ \nu_{vv} = \frac{1}{\text{det}(G^\dagger G)} \sum_{l=1}^{L_a} \cos^2 \theta_l \geq \frac{1}{L_a} \]

\[ \nu = \nu_{uu} + \nu_{vv} \geq \frac{4}{L_a} \]

- GDOP (unit less quantity)
• No upper limits in LSF vs. upper and lower limits in OI.
• A clear demonstration of uncertainty along the baseline.
Computing time and noise level of vector currents

A month hourly vector current data (May 2004)

(a) Calculation time (seconds) vs. the number of vector grid points

(b) Power spectrum of vector current: $S_{UWLS}(f)$ (gray), $S_{WLS}(f)$ (black), $S_{OI}(f)$ (dashed black)

Noise level: $10.3 \text{ cm/s}$

Calculated speed: $4.3 \text{ cm/s}$
\[ \mathbf{u} = \mathbf{u}_\phi + \mathbf{u}_\psi = \nabla_H \phi + \mathbf{k} \times \nabla_H \psi, \]

\[ \mathbf{d}(x) = \sum_k \mathbf{m}(k) \exp(i \mathbf{k} \cdot \mathbf{x}) = \mathbf{Gm}. \]

If the covariance matrix is stationary,

\[ \langle \mathbf{d}(x_1) \mathbf{d}(x_2)^\dagger \rangle = \text{cov}(x_1 - x_2), \]

\[ \langle \mathbf{m}(\mathbf{k}_1) \mathbf{m}(\mathbf{k}_2)^\dagger \rangle = \sigma^2(\mathbf{k}_1) \delta(\mathbf{k}_1 - \mathbf{k}_2), \]

\[ \text{cov}(\Delta \mathbf{x}) = \sum_k \sigma^2(\mathbf{k}) \exp(i \mathbf{k} \cdot \Delta \mathbf{x}) = \mathbf{G} \langle \mathbf{m} \mathbf{m}^\dagger \rangle, \]

Spatial covariance is equivalent to the Fourier transformed wavenumber spectra

\[ \text{cov}_{uu}(\Delta \mathbf{x}) \leftrightarrow k^2 S_{\phi\phi}(k) \]

\[ S_{\phi\phi}(k) \leftrightarrow \text{cov}_{\phi\phi}(\Delta \mathbf{x}) \]
Summary

- Optimal interpolation in mapping of HF radar-derived radial velocity maps into a vector current map can:
  - Minimize the baseline inconsistency
  - Introduce a unified uncertainty
  - Estimate the dynamic quantities (e.g. stream function, velocity potential)
  - Detect outliers in the radial velocities.

- MATLAB scripts for vector current mapping using OI are available upon request. (syongkim@kaist.ac.kr)
BACKUP SLIDES
Formulation (LSF vs. OI)

Least-squares-fit (LSF)

\[ d = G_a m_a + e \]
\[ d_1 = u \cos \theta_1 + v \sin \theta_1 + e_1 \]
\[ d_2 = u \cos \theta_2 + v \sin \theta_2 + e_2 \]
\[ d_3 = u \cos \theta_3 + v \sin \theta_3 + e_3 \]

\[ \hat{m} = (G_a^T G_a)^{-1} G_a^T d \]

Optimal interpolation (OI)

\[ d = G m \]
\[ d_1 = u_1 \cos \theta_1 + v_1 \sin \theta_1 \]
\[ d_2 = u_2 \cos \theta_2 + v_2 \sin \theta_2 \]
\[ d_3 = u_3 \cos \theta_3 + v_3 \sin \theta_3 \]

\[ \hat{m} = P G^T (G P G^T + R)^{-1} d \]
\[ = (G^T R^{-1} G + P^{-1})^{-1} G^T R^{-1} d \]

\( P = \text{a priori model covariance} \)
\( R = \text{observational error covariance} \)

(Kim et al, JGR 2008)