# Revisit:
# Optimal interpolation in mapping of high-frequency radar-derived surface radial velocity maps

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Eric Terrill and Bruce Cornuelle (SIO)

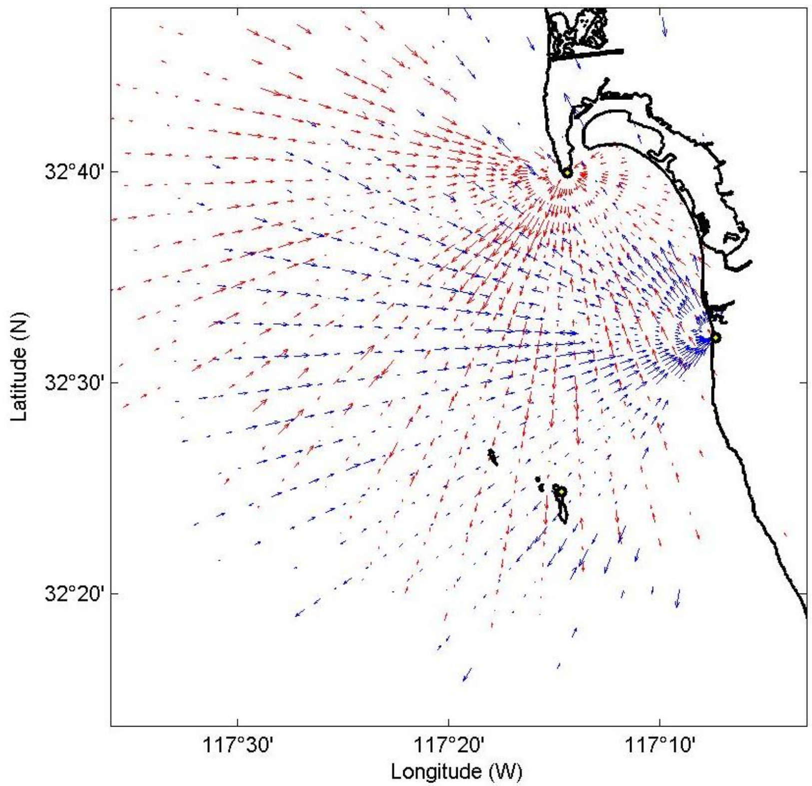
Kim *et al* (2008), Mapping surface currents from HF radar radial velocity measurements using optimal interpolation, *J. Geophys. Res. Oceans* 113, C10023, doi:10.1029/2007JC004244

Kim, S. Y., (2010) Observations of submesoscale eddies using high-frequency radar-derived kinematic and dynamic quantities, *Cont. Shelf Res.* 30(15), 1639 - 1655, doi:10.1016/j.csr.2010.06.011

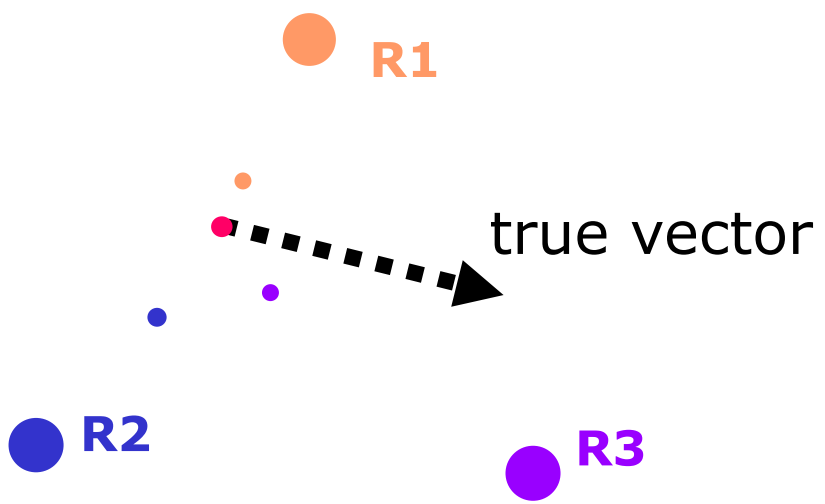
# Outline

- Formulation of least-squares fit (LSF) and optimal interpolation (OI) via inverse method.
- Comparison of correlation functions, vector current maps, uncertainty ellipses
- Summary

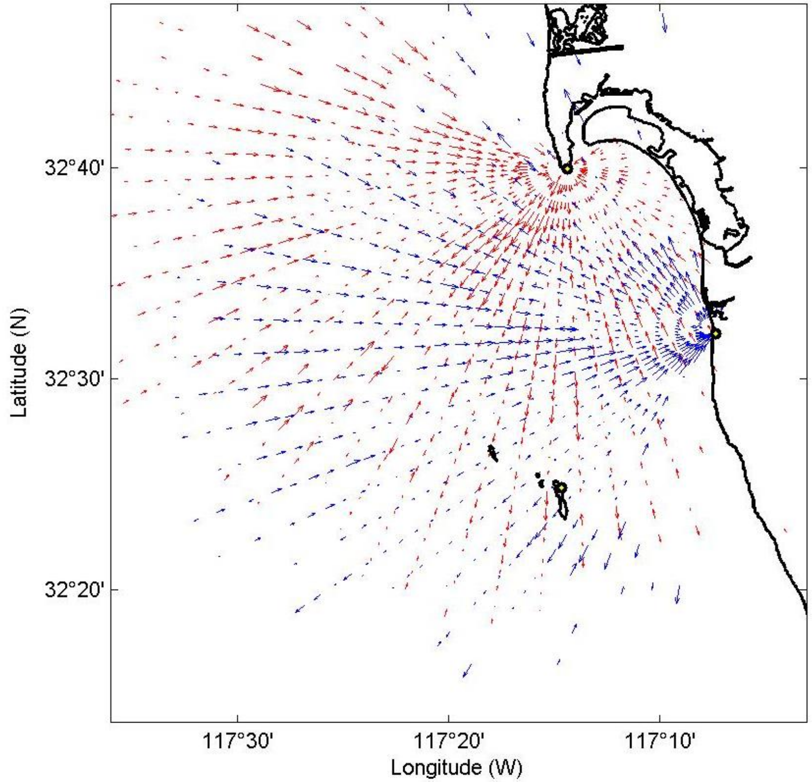
# Radial velocity maps




- Radial velocity maps
  - Range
  - Bearing angle
  - Scalar velocity



# Radial velocity maps

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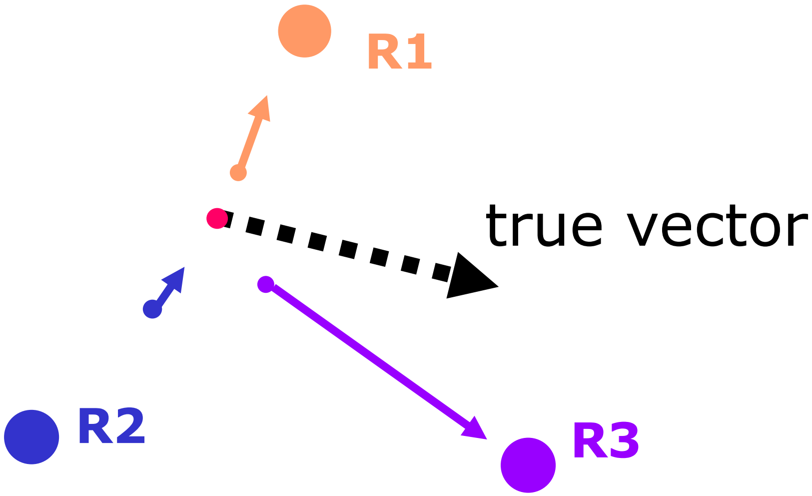
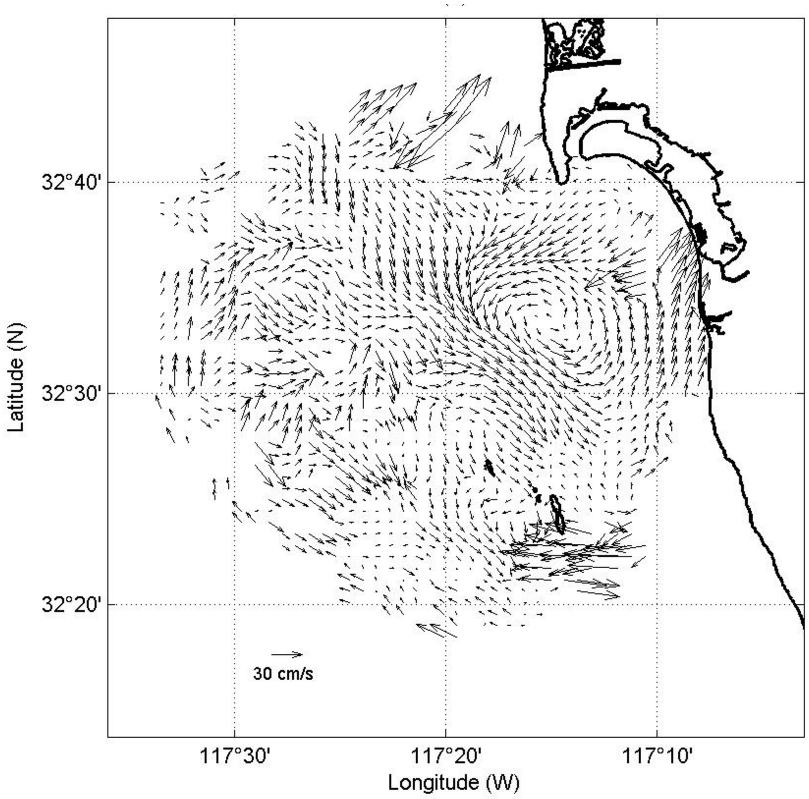
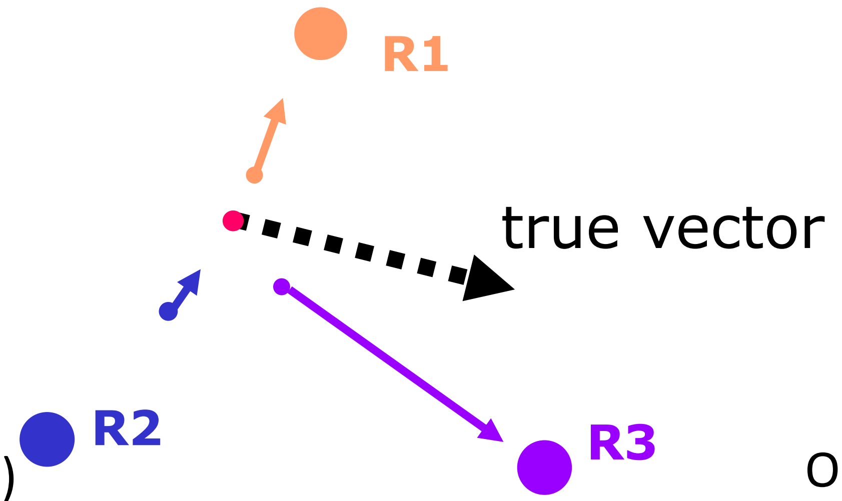


# Vector current maps




- Least-squares fit
  - Best fit of u and v with available radials within a search radius
- Spatially and temporally noisy data due to outliers
- Baseline inconsistency
- Divergence and vorticity should be estimated from vector current maps

# Formulation (LSF vs. OI)




Least-squares-fit (LSF)

$$d = G_a m_a + e$$

$$d_1 = u \cos\theta_1 + v \sin\theta_1 + e_1$$
$$d_2 = u \cos\theta_2 + v \sin\theta_2 + e_2$$
$$d_3 = u \cos\theta_3 + v \sin\theta_3 + e_3$$

$$\hat{m} = (G_a^T G_a)^{-1} G_a^T d$$

Optimal interpolation (OI)

$$d = Gm$$

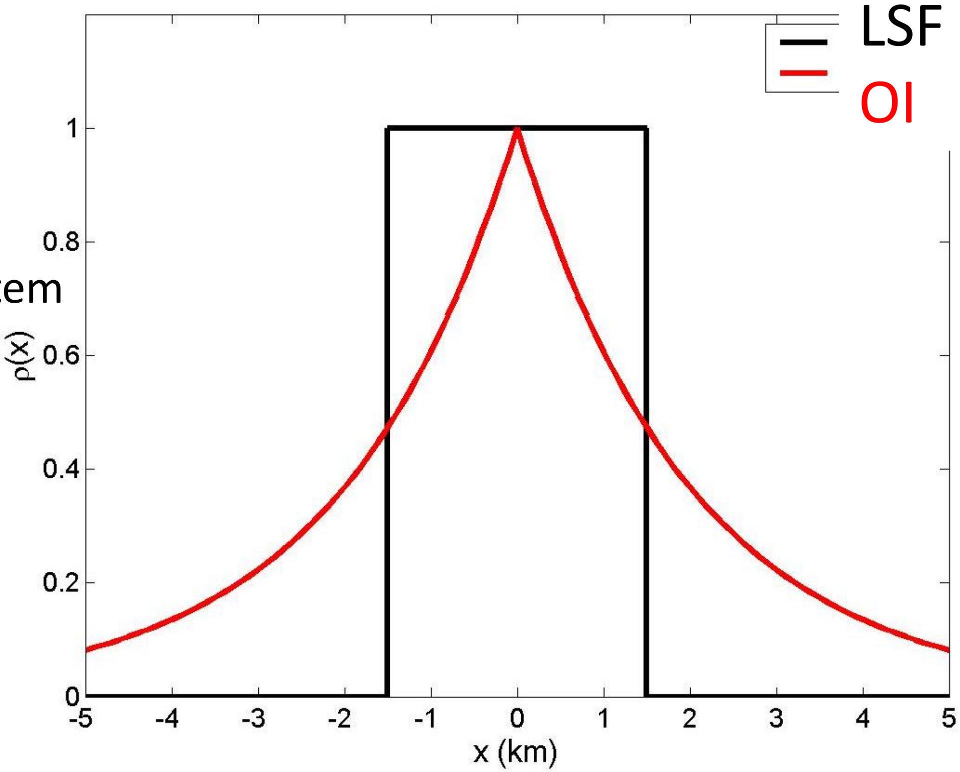
$$d_1 = u_1 \cos\theta_1 + v_1 \sin\theta_1$$
$$d_2 = u_2 \cos\theta_2 + v_2 \sin\theta_2$$
$$d_3 = u_3 \cos\theta_3 + v_3 \sin\theta_3$$

$$\hat{m} = PG^T(GPG^T + R)^{-1} d$$
$$= (G^T R^{-1} G + P^{-1})^{-1} G^T R^{-1} d$$

P = *a prior* model covariance
R = observational error covariance

(Kim et al, JGR 2008)

# Correlation functions (LSF vs. OI)

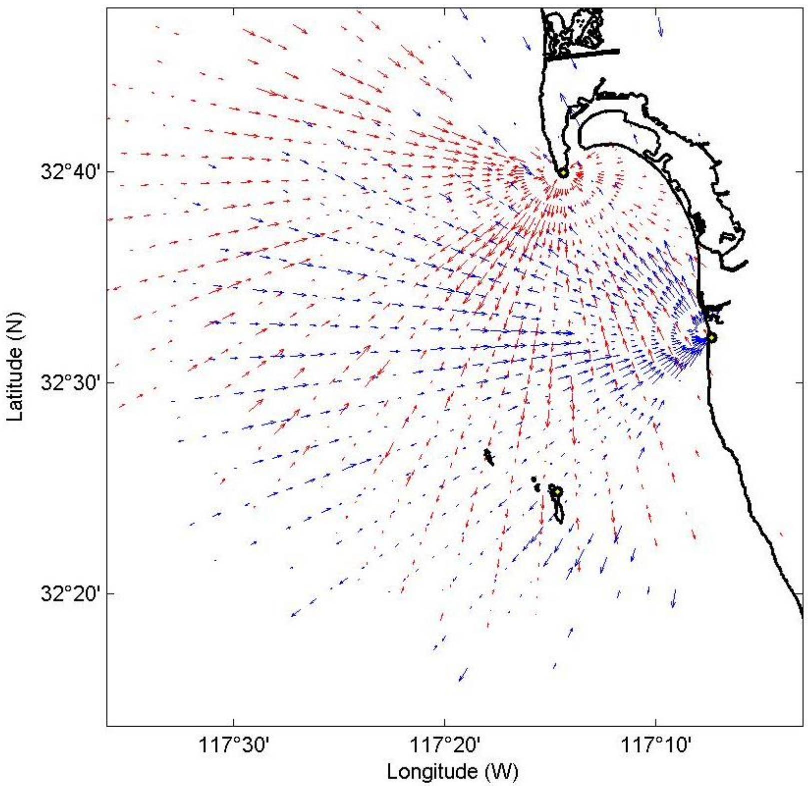


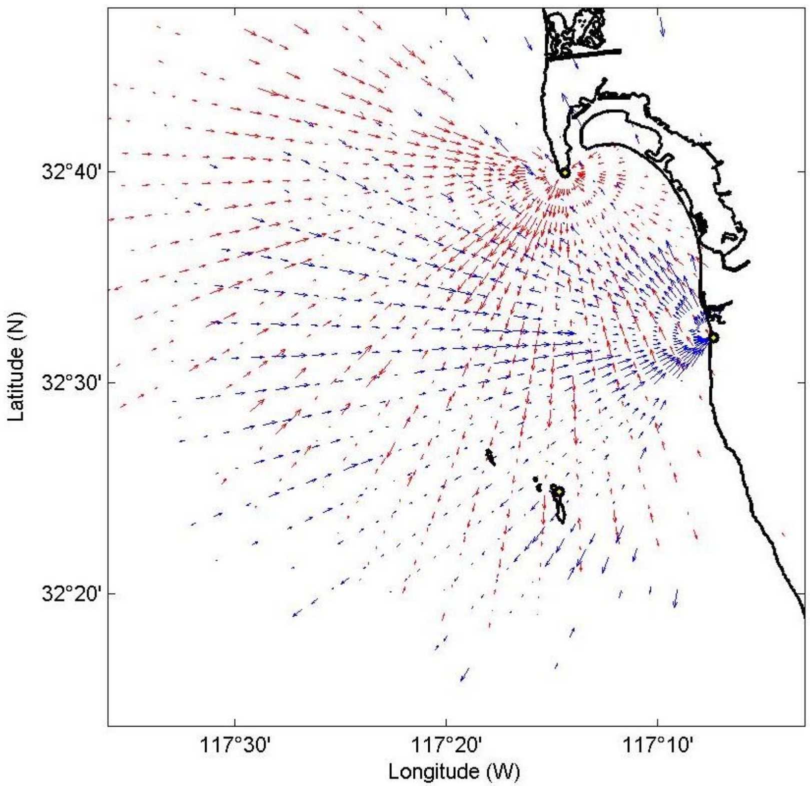

Examples of 25 MHz system with 1 km vector grid

- Constant velocity within a search range: segmented correlation function
- An example of 1.5 km search radius

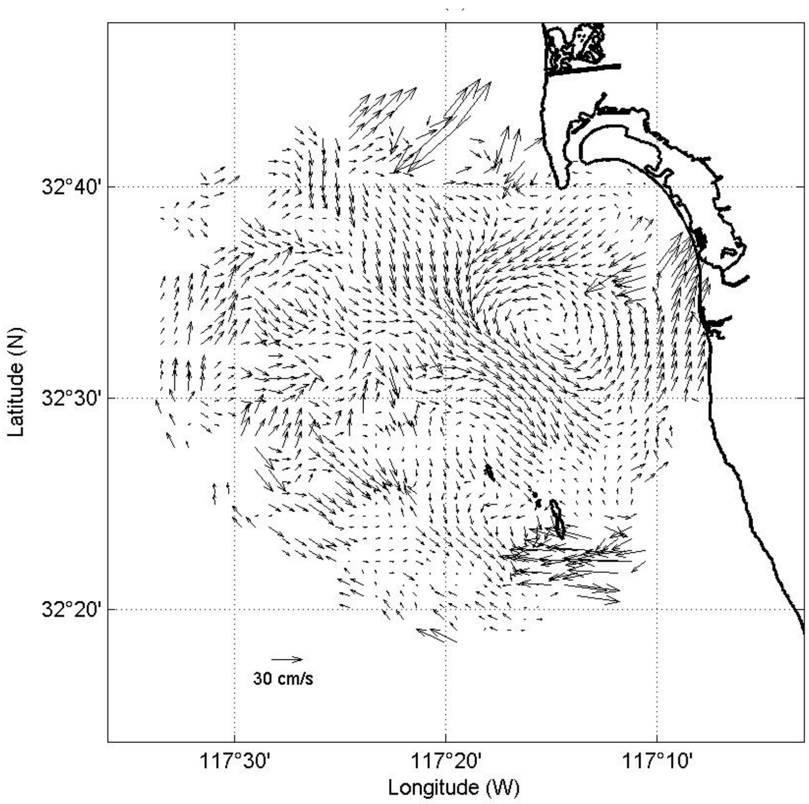
- Spatial weighting by distance: exponential, Gaussian, or observed covariance function
- An example of an exponential corr. function with 2 km decorrleation length scale

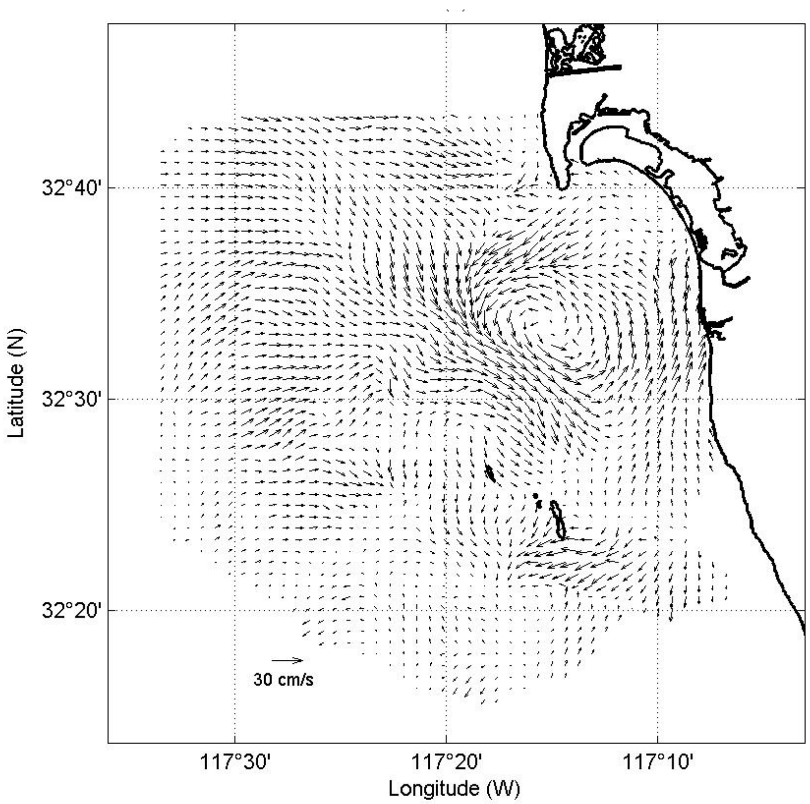
# Radial velocity maps



# Vector current maps (LSF vs. OI)




- Spatially smooth vector current field
- Improved baseline inconsistency
- A search radius in OI is adopted to reduce the calculation time.

# Uncertainty (LSF vs. OI)

## Uncertainty estimate (*a posterior* error covariance)

$$\hat{P} = (G_a^T G_a)^{-1}$$

$$\hat{P} = (G^T R^{-1} G + P^{-1})^{-1}$$

## Uncertainty normalized with error variance

$$\varphi = (G_a^T G_a)^{-1}$$

$$\varphi = (G^T G + P^{-1} \sigma_r^2 I)^{-1}$$

## Uncertainty normalized with model variance

?

$$\hat{X} = P^{-1/2} \hat{P} P^{-1/2}$$

$P$ = *a prior* model covariance = $\sigma_s^2 I$
$R$ = observational error covariance = $\sigma_r^2 I$

# Uncertainty (LSF vs. OI)

$$\boldsymbol{\nu} = \left(\mathbf{G}^{\dagger}\mathbf{G}\right)^{-1} = \begin{bmatrix} \nu_{uu} & \nu_{uv} \\ \nu_{vu} & \nu_{vv} \end{bmatrix}$$

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_{uu} & \kappa_{uv} \\ \kappa_{vu} & \kappa_{vv} \end{bmatrix}$$

$$\nu_{uu} = \frac{1}{\det\left(\mathbf{G}^{\dagger}\mathbf{G}\right)} \sum_{l=1}^{L_a} \sin^2\theta_l \geq \frac{1}{L_a}$$

$$\nu_{vv} = \frac{1}{\det\left(\mathbf{G}^{\dagger}\mathbf{G}\right)} \sum_{l=1}^{L_a} \cos^2\theta_l \geq \frac{1}{L_a}$$

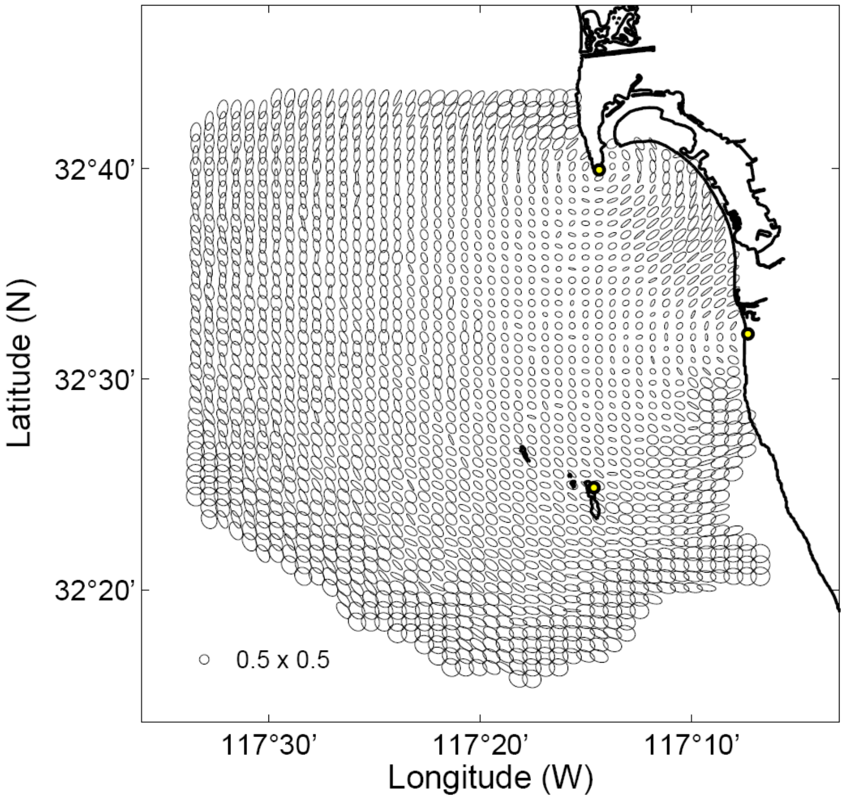
$$0 \leq \kappa_{uu} \leq \gamma^2$$

$$0 \leq \kappa_{vv} \leq \gamma^2$$

$$\nu = \nu_{uu} + \nu_{vv} \geq \frac{4}{L_a}$$

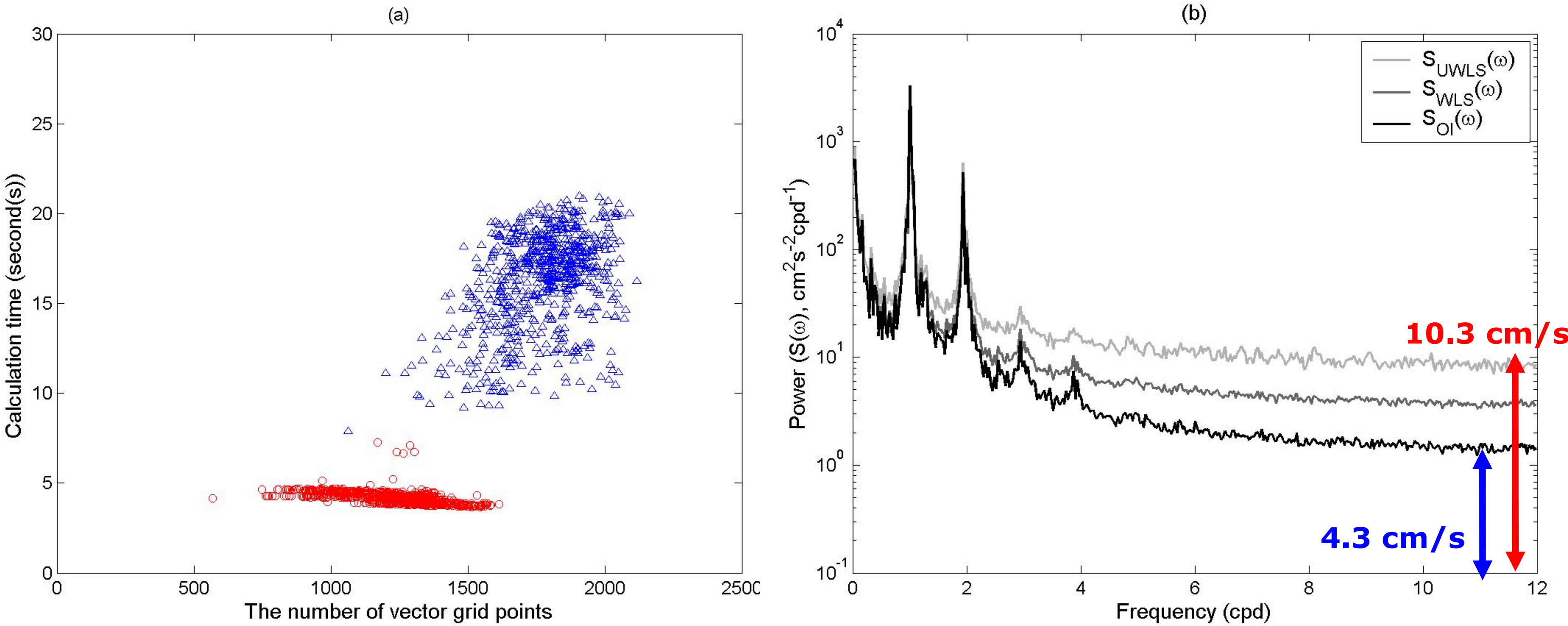
- GDOP (unit less quantity)

# Uncertainty ellipses




- No upper limits in LSF vs. upper and lower limits in OI.
- A clear demonstration of uncertainty along the baseline.

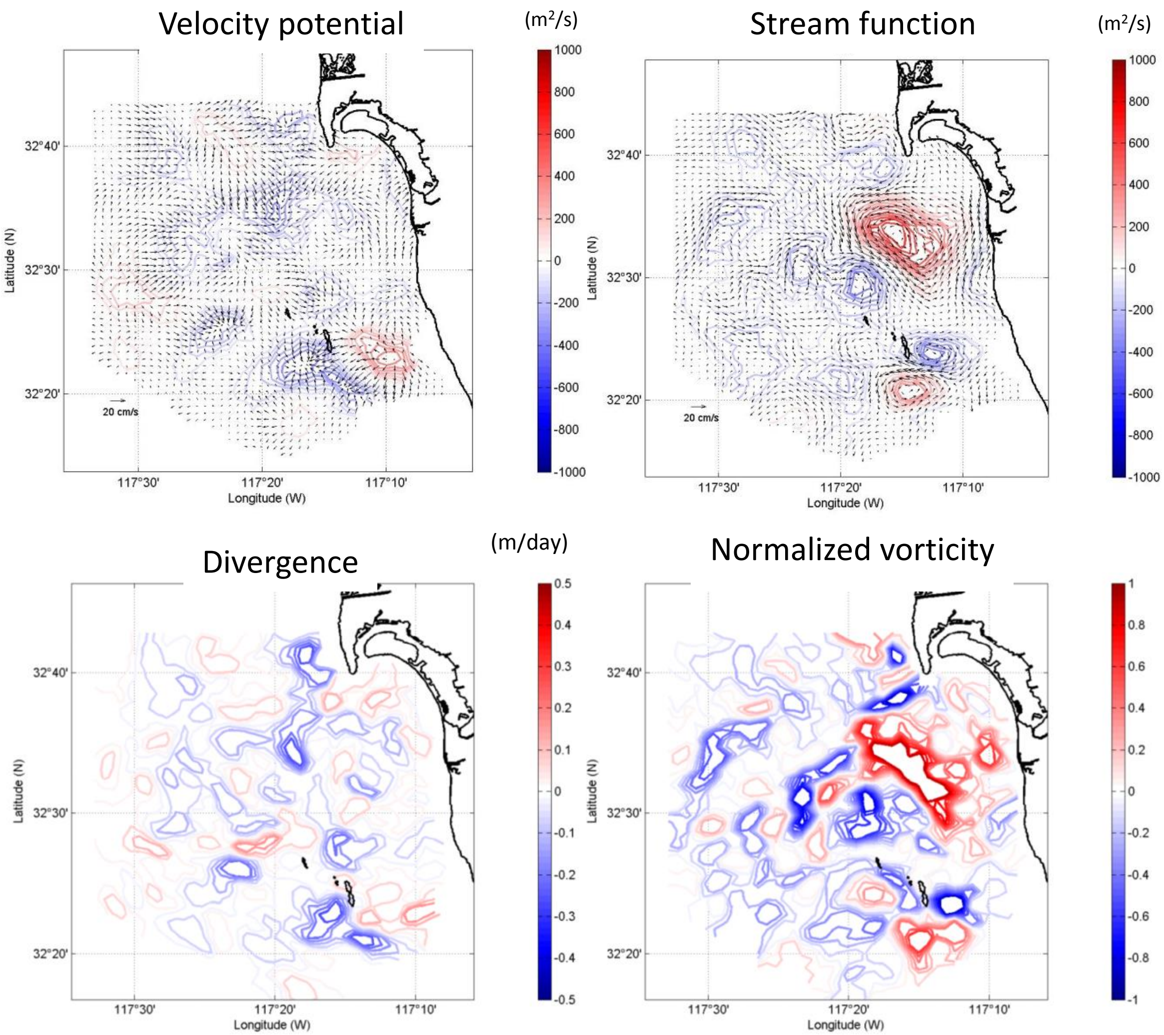
# Computing time and noise level of vector currents




A month hourly vector current data (May 2004)

# Kinematic and dynamic quantities

$$\mathbf{u} = \mathbf{u}_\phi + \mathbf{u}_\psi = \nabla_H \phi + \mathbf{k} \times \nabla_H \psi,$$




$$\mathbf{d}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{m}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) = \mathbf{G}\mathbf{m}.$$

If the covariance matrix is stationary,

$$\langle \mathbf{d}(\mathbf{x}_1)\mathbf{d}(\mathbf{x}_2)^\dagger \rangle = \mathrm{cov}(\mathbf{x}_1 - \mathbf{x}_2),$$

$$\langle \mathbf{m}(\mathbf{k}_1)\mathbf{m}(\mathbf{k}_2)^\dagger \rangle = \sigma^2(\mathbf{k}_1)\delta(\mathbf{k}_1 - \mathbf{k}_2),$$

$$\mathrm{cov}(\Delta\mathbf{x}) = \sum_{\mathbf{k}} \sigma^2(\mathbf{k}) \exp(i\mathbf{k} \cdot \Delta\mathbf{x}) = \mathbf{G}\langle \mathbf{m}\mathbf{m}^\dagger \rangle,$$

Spatial covariance
is equivalent to
the Fourier transformed
wavenumber spectra

$\mathrm{cov}_{uu}(\Delta x) \leftrightarrow k^2 S_{\phi\phi}(k)$
$S_{\phi\phi}(k) \leftrightarrow \mathrm{cov}_{\phi\phi}(\Delta x)$

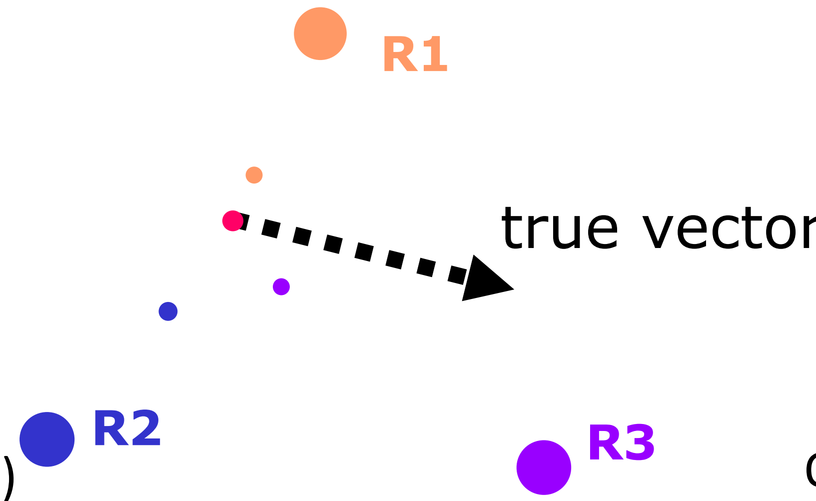
# Summary

- Optimal interpolation in mapping of HF radar-derived radial velocity maps into a vector current map can
  - Minimize the baseline inconsistency
  - Introduce a unified uncertainty
  - Estimate the dynamic quantities (e.g. stream function, velocity potential)
  - Detect outliers in the radial velocities.

- MATLAB scripts for vector current mapping using OI are available up on request. (syongkim@kaist.ac.kr)

# BACKUP SLIDES

# Formulation (LSF vs. OI)

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