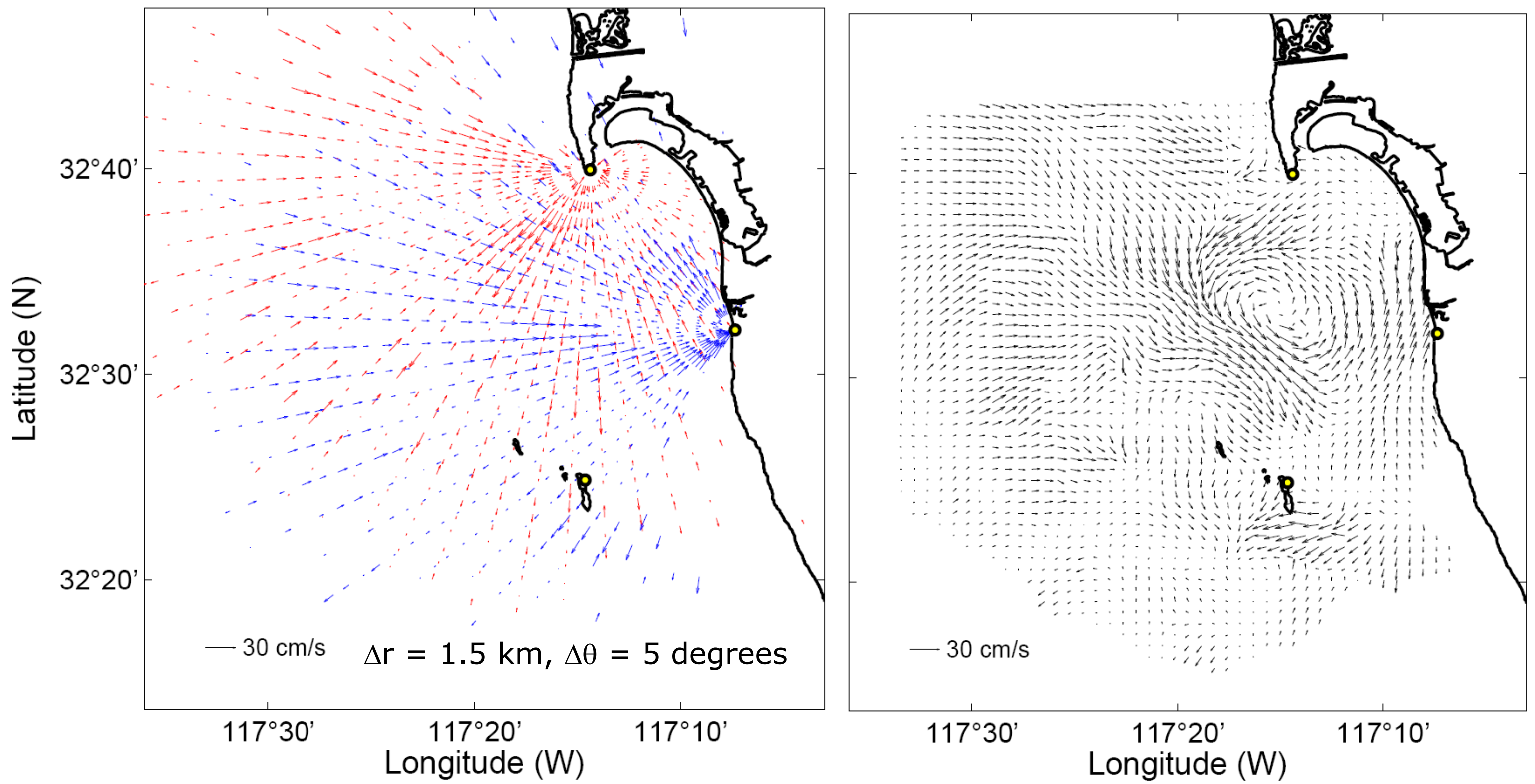
A direct estimate of horizontal spatial covariance from non-orthogonally sampled surface velocities



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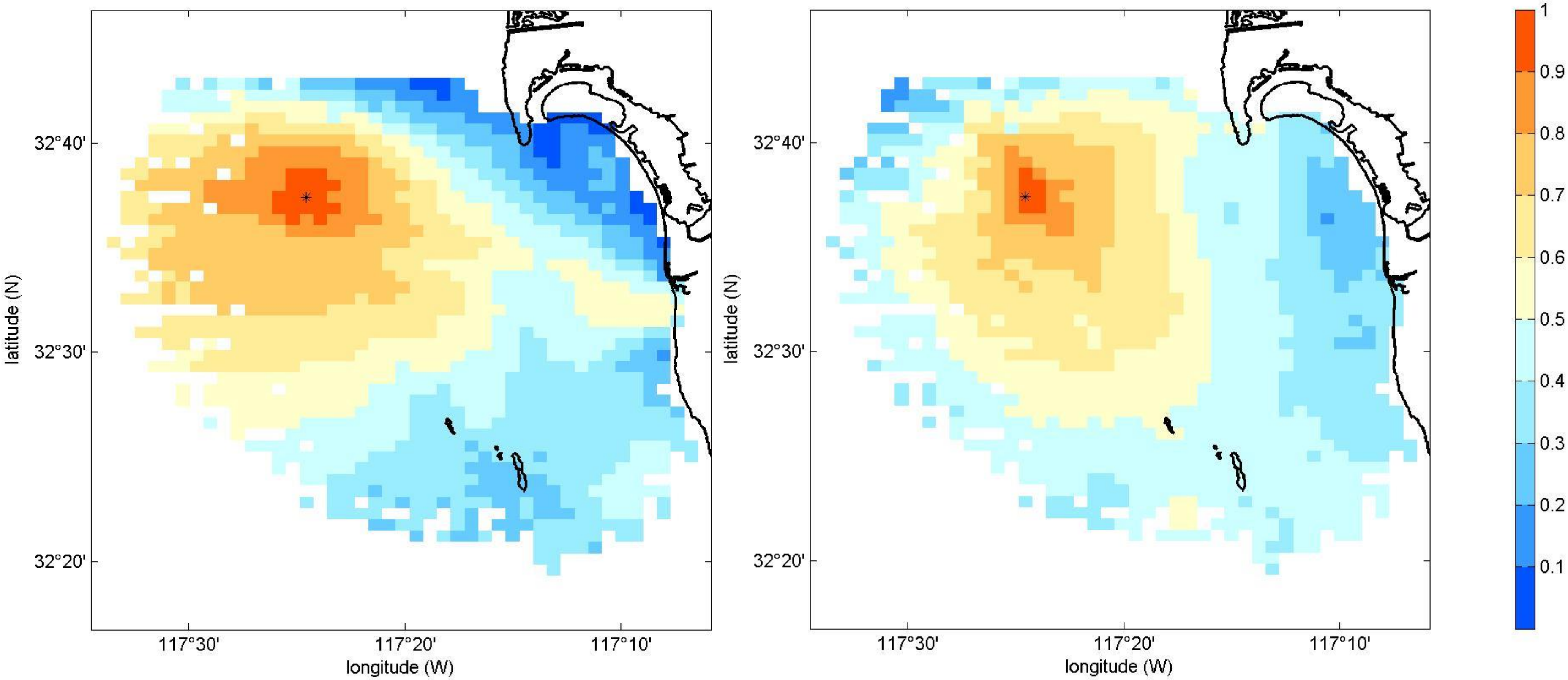
A direct estimate of horizontal spatial covariance from non-orthogonally sampled surface velocities

$\rho_{uu}(\mathbf{x}, \mathbf{y}, x', y')$ **offshore** $\rho_{vv}(\mathbf{x}, \mathbf{y}, x', y')$



- Correlations of the 2-year vector current data

(Kim *et al*. JGR, 2007)

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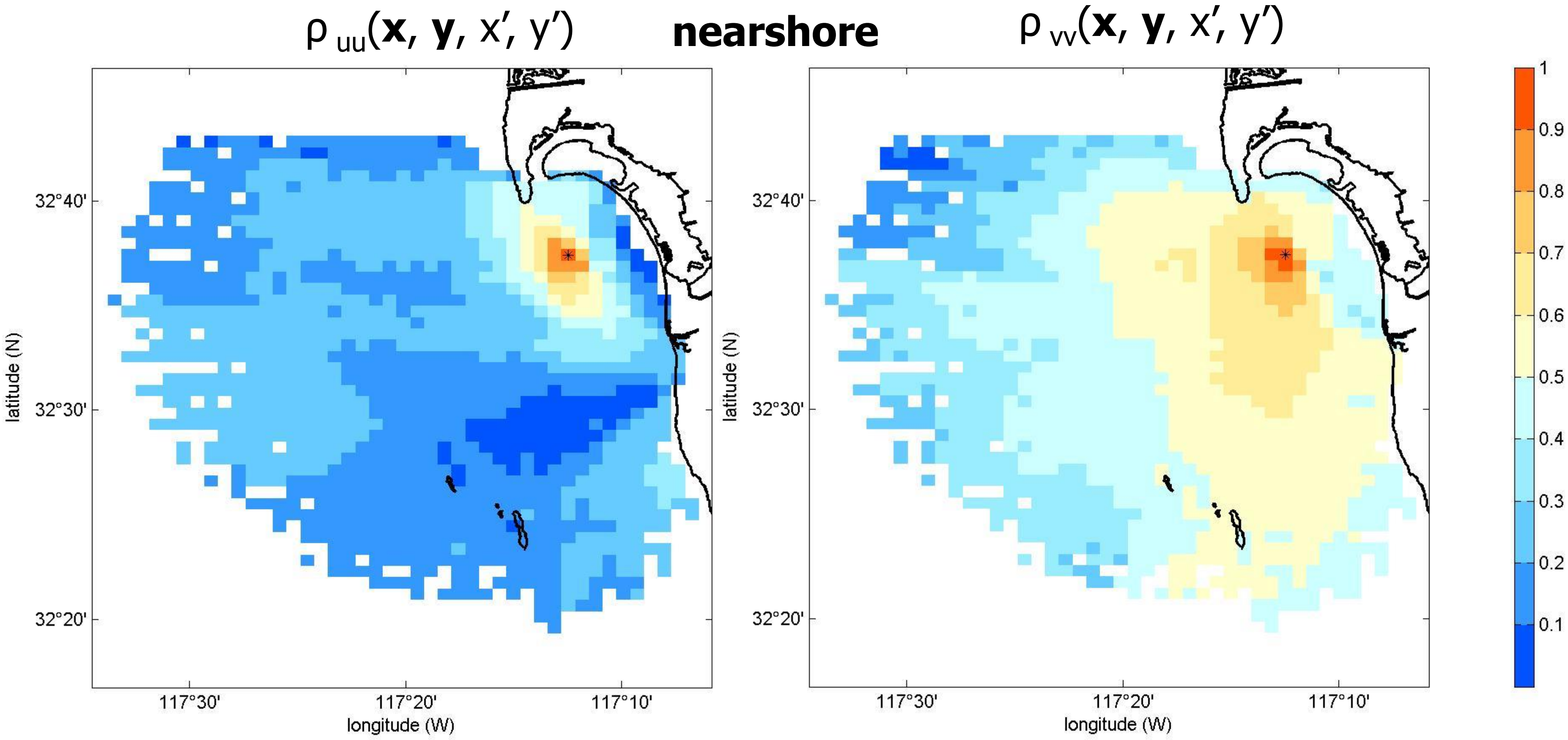
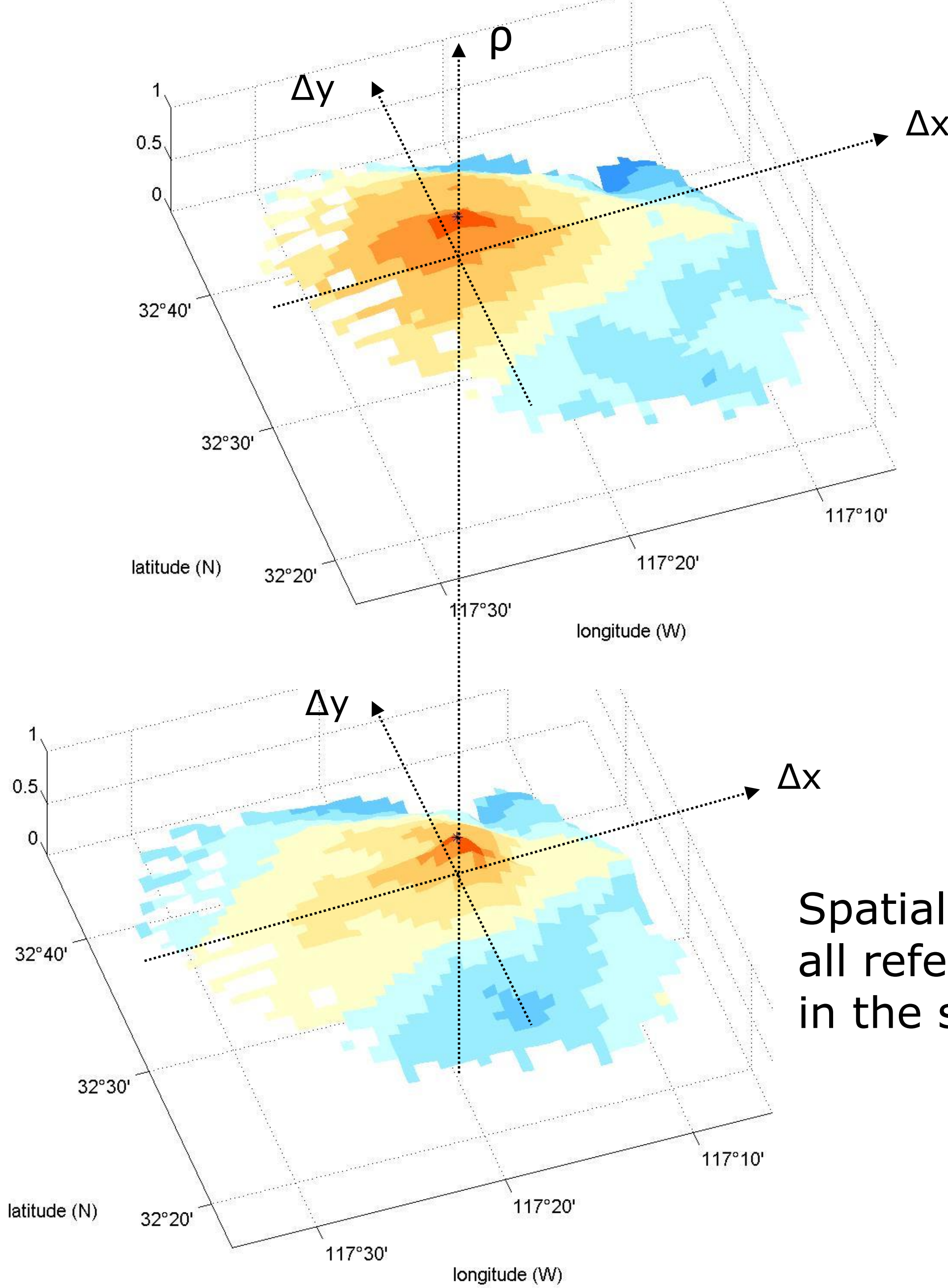


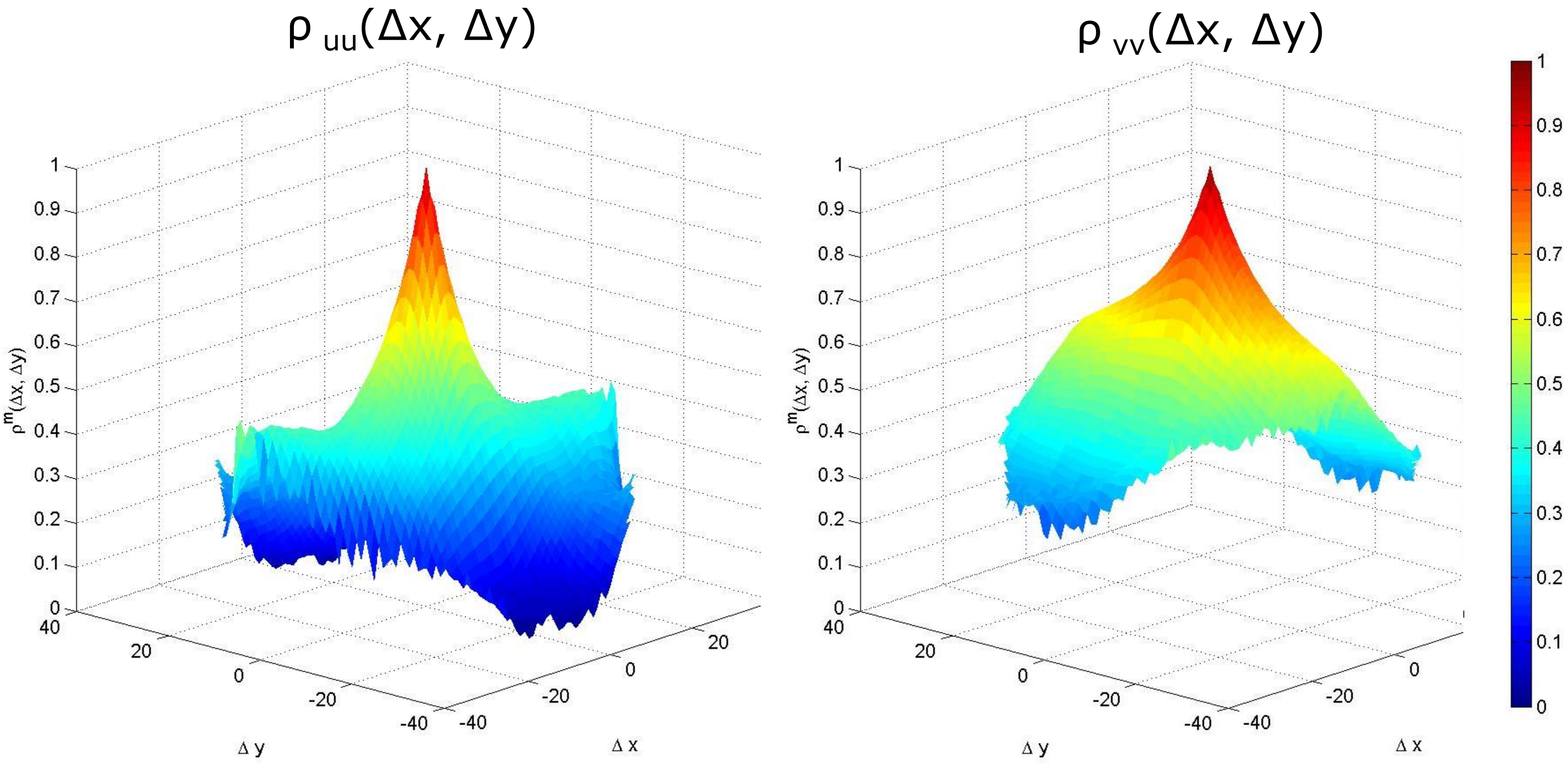
- De-correlation scale is the function of space (x, y).

(Kim *et al*. JGR, 2007)



Spatial lag average over all reference grid points in the study domain

A direct estimate of horizontal spatial covariance from non-orthogonally sampled surface velocities

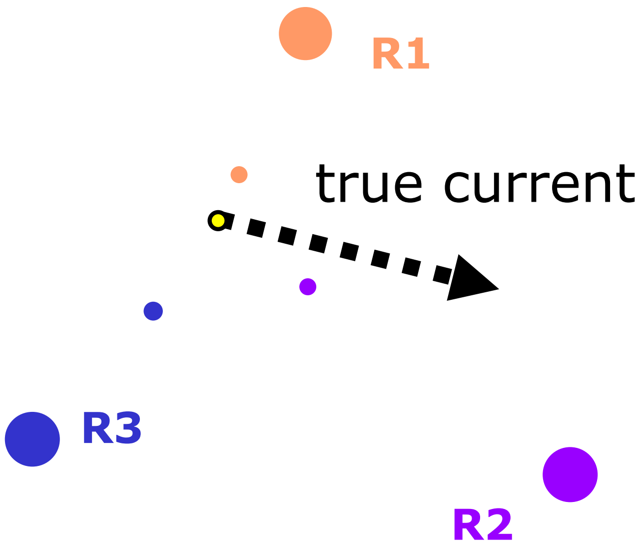


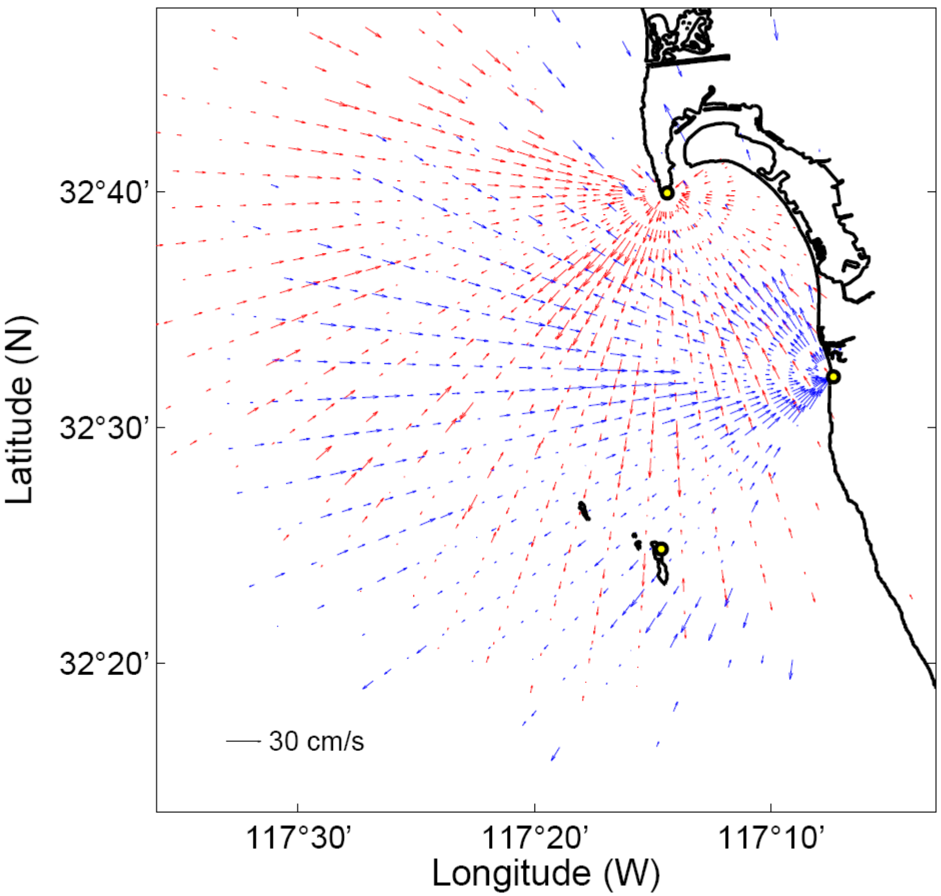
- Spatially composite correlation over the study domain
- Exponential shape (not Gaussian)

(Kim *et al*. JGR, 2007)

A direct estimate of horizontal spatial covariance from non-orthogonally sampled surface velocities

$$r(t) = \mathbf{g}^{\dagger}\mathbf{w}(t) + \epsilon(t) = u(t)\cos\theta + v(t)\sin\theta + \epsilon(t),$$





A direct estimate of horizontal spatial covariance from non-orthogonally sampled surface velocities

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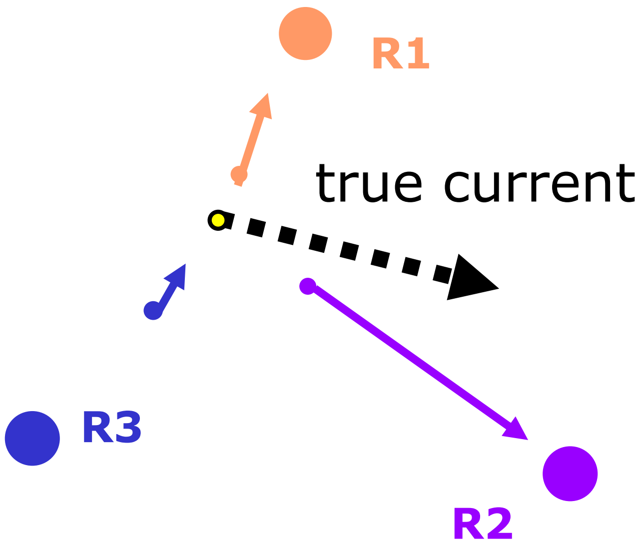

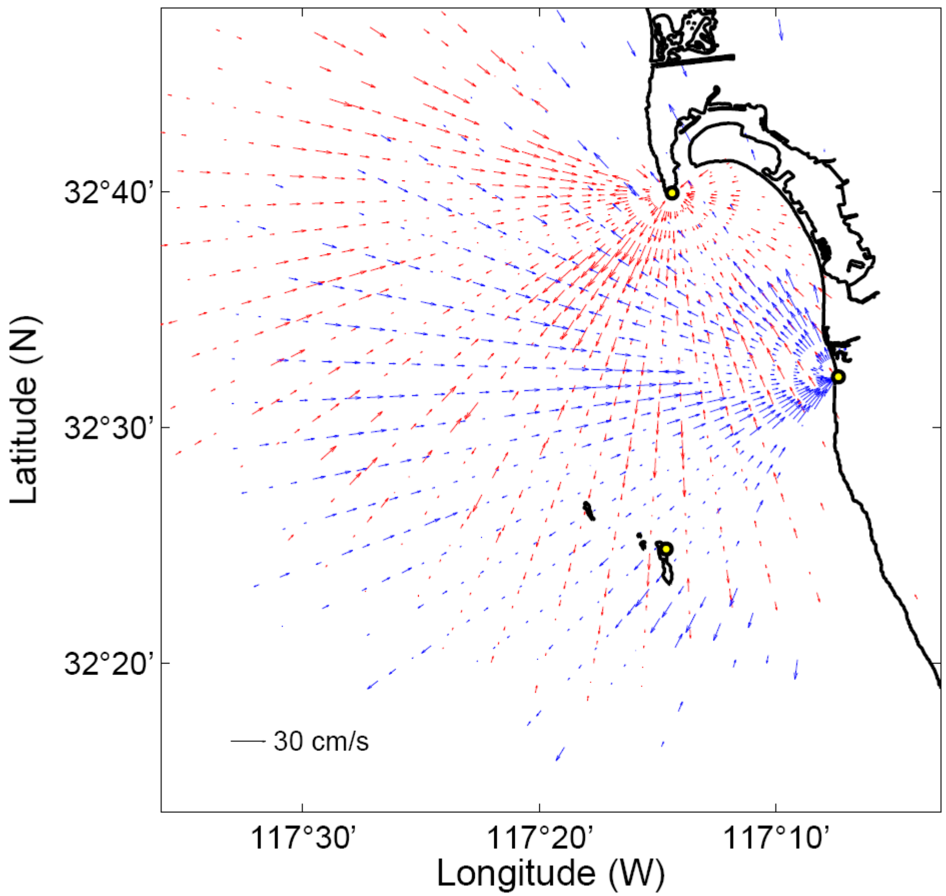

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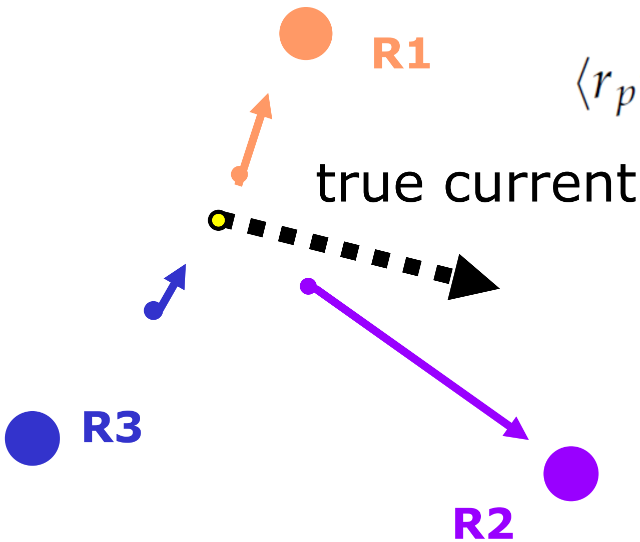

$$\langle r_p r_q^{\dagger}\rangle = \mathbf{g}_p^{\dagger}\langle \mathbf{w}_p \mathbf{w}_q^{\dagger}\rangle \mathbf{g}_q, \quad (11)$$

$$= \cos\theta_p \cos\theta_q \langle u_p u_q^{\dagger}\rangle + \cos\theta_p \sin\theta_q \langle u_p v_q^{\dagger}\rangle$$

$$+ \sin\theta_p \cos\theta_q \langle v_p u_q^{\dagger}\rangle + \sin\theta_p \sin\theta_q \langle v_p v_q^{\dagger}\rangle + \langle \epsilon_p \epsilon_q^{\dagger}\rangle.$$

$$\mathrm{vec}\left(\langle \mathbf{r}\mathbf{r}^{\dagger}\rangle\right) = \alpha \mathrm{vec}\left(\langle \mathbf{u}\mathbf{u}^{\dagger}\rangle\right) + \beta \mathrm{vec}\left(\langle \mathbf{u}\mathbf{v}^{\dagger}\rangle\right) + \gamma \mathrm{vec}\left(\langle \mathbf{v}\mathbf{u}^{\dagger}\rangle\right) + \delta \mathrm{vec}\left(\langle \mathbf{v}\mathbf{v}^{\dagger}\rangle\right) + \mathrm{vec}\left(\langle \epsilon\epsilon^{\dagger}\rangle\right), \quad (13)$$

$$\mathbf{d} = \mathbf{G}\mathbf{m} + \mathbf{e}, \quad (14)$$

A direct estimate of horizontal spatial covariance from non-orthogonally sampled surface velocities

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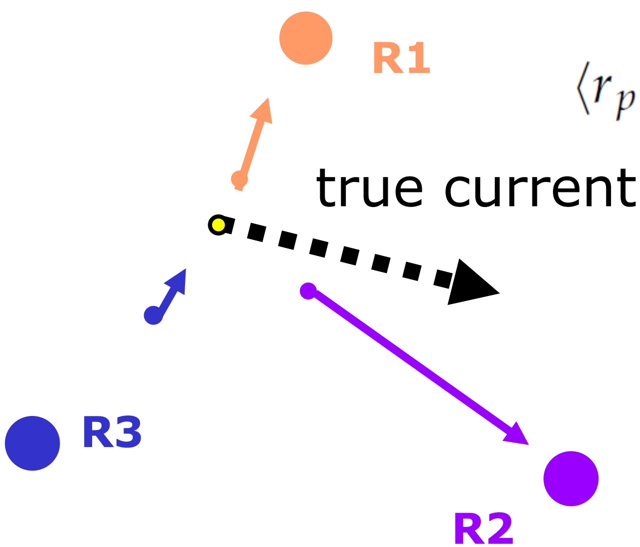

$$\langle r_p r_q^\dagger \rangle = \mathbf{g}_p^\dagger \langle \mathbf{w}_p \mathbf{w}_q^\dagger \rangle \mathbf{g}_q, \tag{11}$$

$$= \cos\theta_p \cos\theta_q \langle u_p u_q^\dagger \rangle + \cos\theta_p \sin\theta_q \langle u_p v_q^\dagger \rangle$$

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$$\mathrm{vec}\left(\langle \mathbf{r}\mathbf{r}^\dagger \rangle\right) = \alpha \mathrm{vec}\left(\langle \mathbf{u}\mathbf{u}^\dagger \rangle\right) + \beta \mathrm{vec}\left(\langle \mathbf{u}\mathbf{v}^\dagger \rangle\right)$$

$$+ \gamma \mathrm{vec}\left(\langle \mathbf{v}\mathbf{u}^\dagger \rangle\right) + \delta \mathrm{vec}\left(\langle \mathbf{v}\mathbf{v}^\dagger \rangle\right) + \mathrm{vec}\left(\langle \epsilon\epsilon^\dagger \rangle\right), \tag{13}$$

$$\mathbf{d} = \mathbf{G}\mathbf{m} + \mathbf{e}, \tag{14}$$

$$\langle u(\mathbf{x}_p) u(\mathbf{x}_q)^\dagger \rangle = \sum_{m=-M}^{M} \sum_{n=-N}^{N} A_{mn} e^{i\mathbf{k}_{mn}\Delta\mathbf{x}_{pq}},$$

where

$$\mathbf{k}_{mn} = (k_m, l_n),$$

$$\Delta\mathbf{x}_{pq} = (x_p - x_q, y_p - y_q).$$

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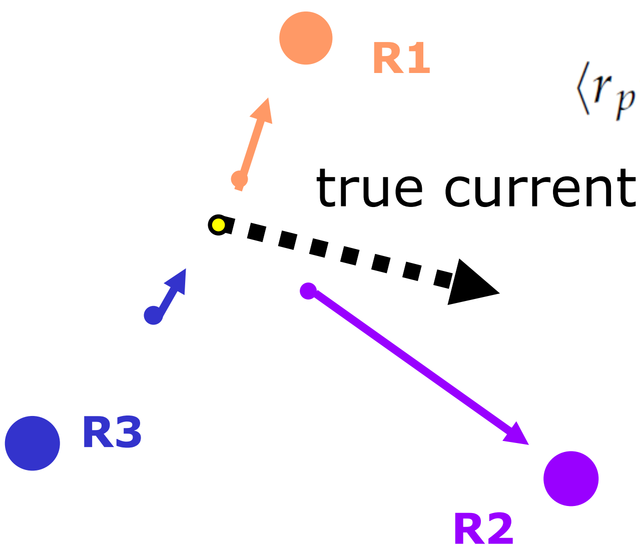

$$\begin{aligned}\langle r_p r_q^{\dagger}\rangle &= \mathbf{g}_p^{\dagger}\langle \mathbf{w}_p \mathbf{w}_q^{\dagger}\rangle \mathbf{g}_q, \\ &= \cos\theta_p\cos\theta_q\langle u_p u_q^{\dagger}\rangle + \cos\theta_p\sin\theta_q\langle u_p v_q^{\dagger}\rangle \\ &+ \sin\theta_p\cos\theta_q\langle v_p u_q^{\dagger}\rangle + \sin\theta_p\sin\theta_q\langle v_p v_q^{\dagger}\rangle + \langle \epsilon_p \epsilon_q^{\dagger}\rangle.\end{aligned} \quad (11)$$

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L=134; d = 9045x1; M = 100; N = 100;

$$\mathbf{d} = \mathbf{G}\mathbf{m} + \mathbf{e}, \quad (14)$$

The number of radial grid points — The number of basis functions in the x and y directions

TABLE 1. Dimensions of the vectors and matrices in the direct estimate of variance and covariance.

Notation	Variance	Covariance
$\mathbf{d}$	$L \times 1$	$L(L+1)/2 \times 1$
$\mathbf{m}$	$3(2M+1)(2N+1) \times 1$	$4(2M+1)(2N+1) \times 1$
$\mathbf{G}$	$L \times 3(2M+1)(2N+1)$	$L(L+1)/2 \times 4(2M+1)(2N+1)$

Motivation

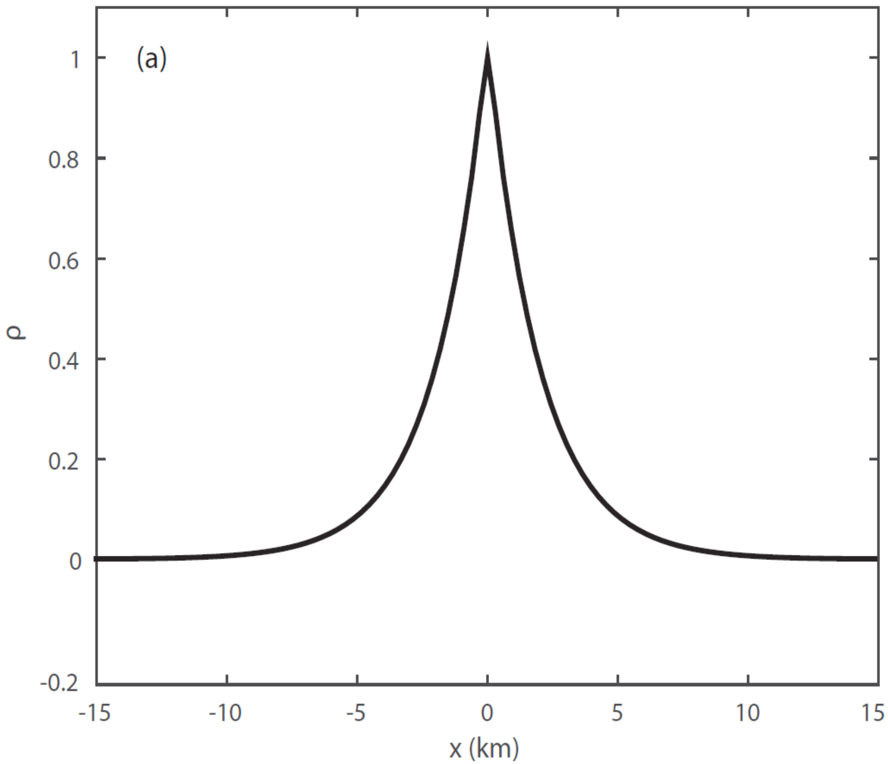
- When we describe the variability of the spatial-temporal data (fields or system) and characterize them, we may examine their covariance structure and decorrelation scales.
- However, the data may not be evenly sampled in space.
- Mapping of data on regularly spaced grid may be required, so the spatial covariance/correlation of the mapped fields contains a bias associated with given assumptions.
  - Correlation: a normalized structure of covariance
- So, how can we directly estimate covariance of unevenly sampled data in space?

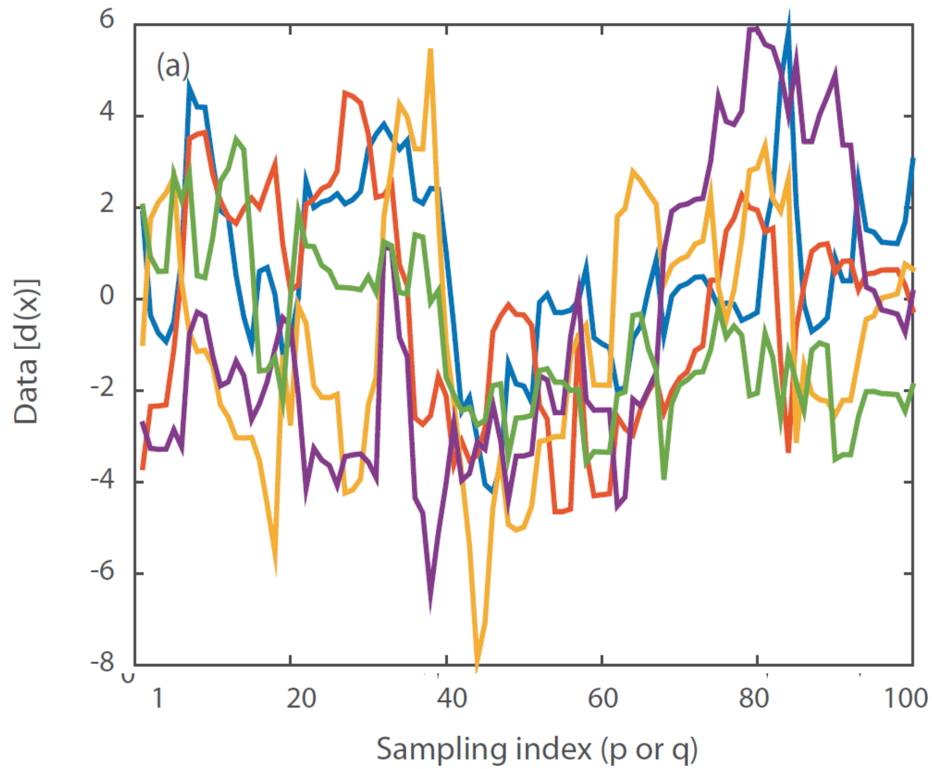
Covariance estimates (1D, scalar)

- Covariance vs. Energy spectra
  - Exponential function with a decorrelation length scale of 2km
  - Unevenly sampled data (plotted on the sampling index, not the physical domain

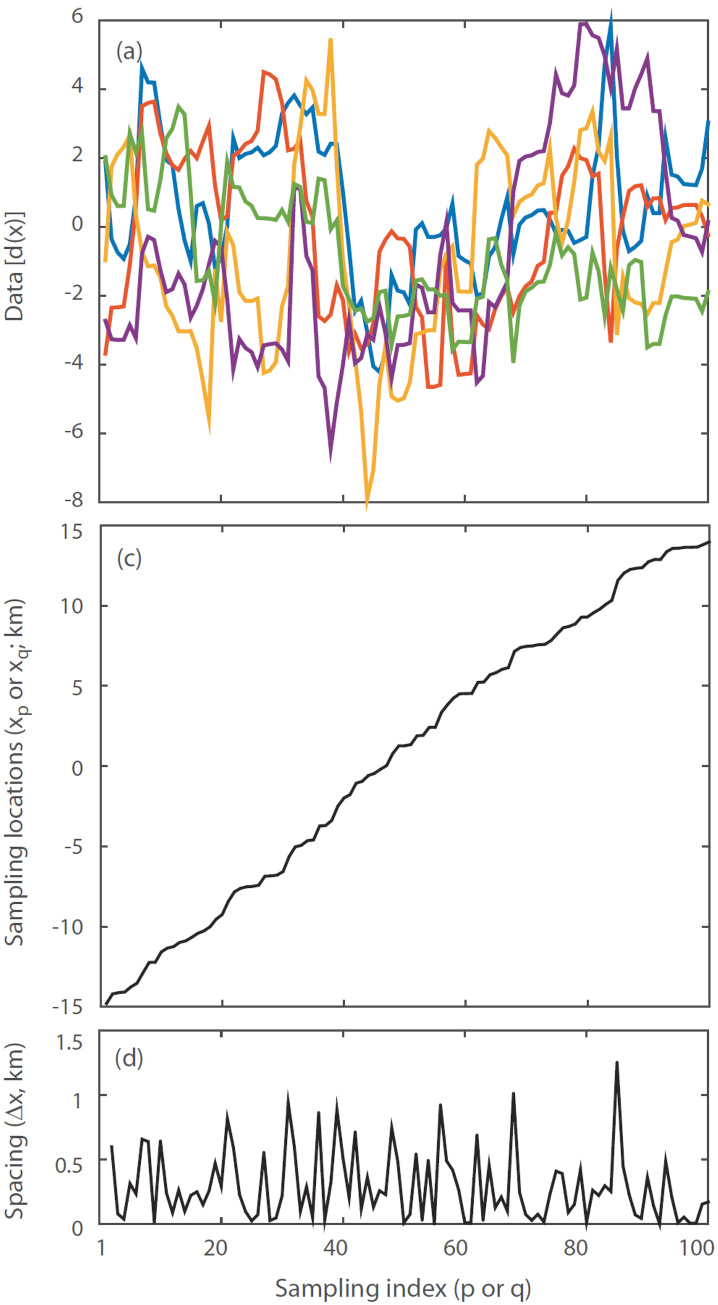
$$\int_{-\infty}^{\infty} e^{-\frac{|x|}{\lambda}} e^{-ikx} \mathrm{d}x = \frac{2\lambda}{1+\lambda^2 k^2}$$

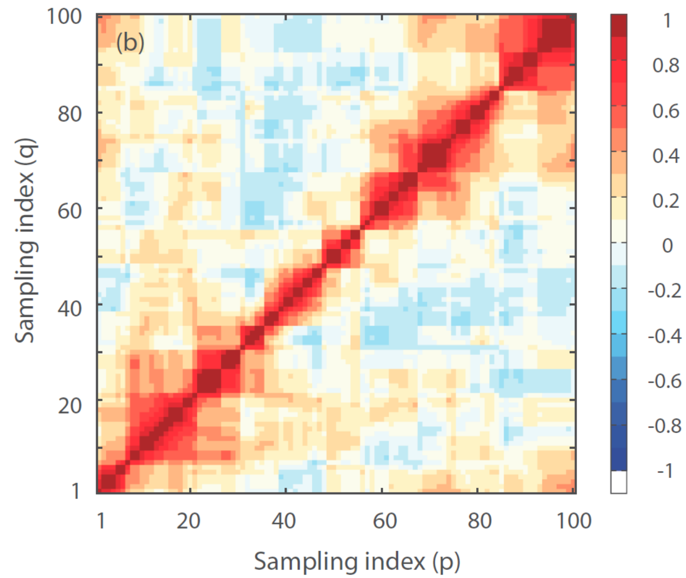
$$d(x,t) = \sum_m a_m(t)\cos(k_m x) + b_m(t)\sin(k_m x),$$
$$= \sum_m \chi_m(t)\phi_m(x),$$





Covariance estimates (1D, scalar)





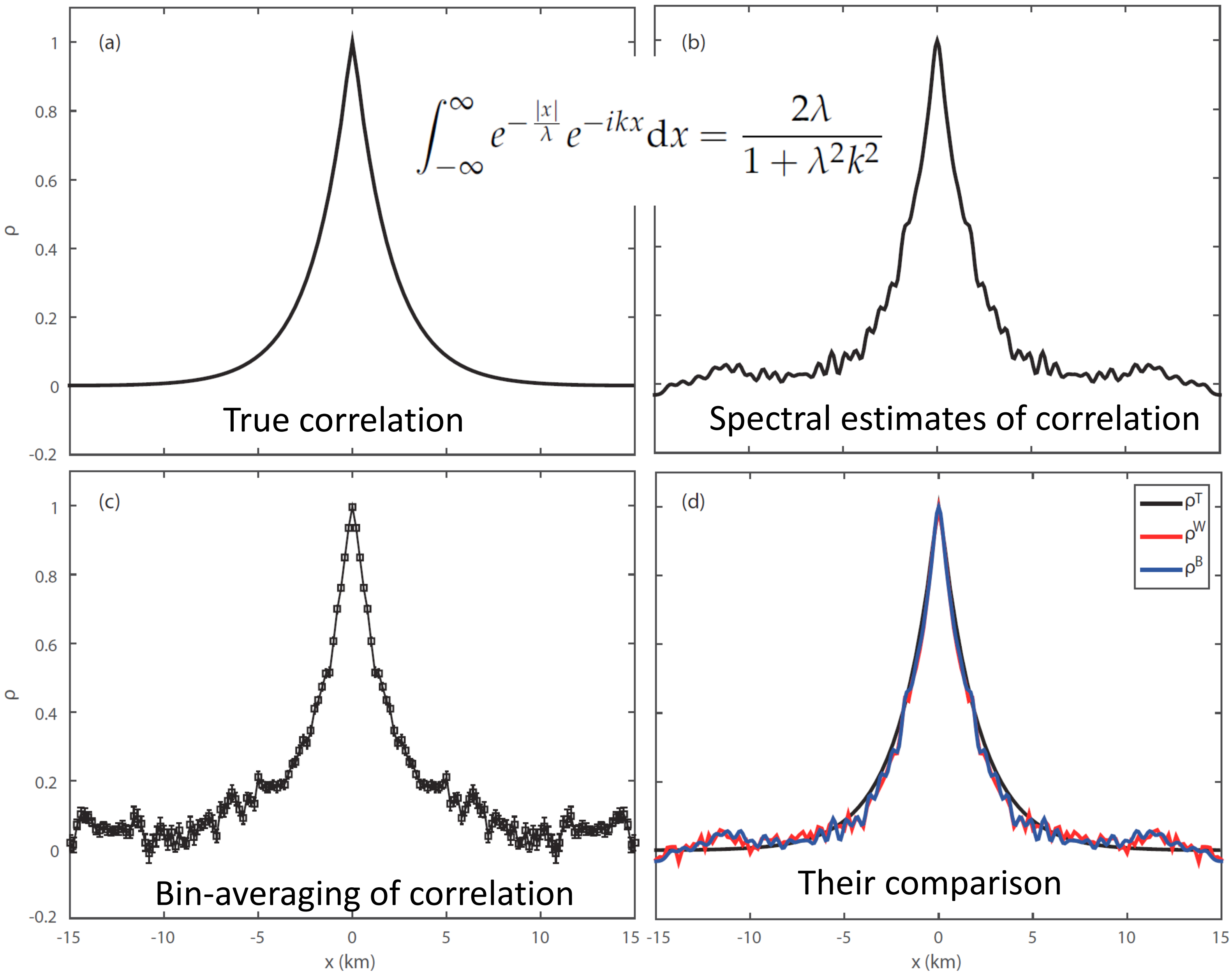
$$\xi_m^2(k) = \xi^2(k_m) = \frac{2\lambda}{1 + k_m^2\lambda^2},$$

$$\begin{aligned} d(x,t) &= \sum_m a_m(t)\cos(k_m x) + b_m(t)\sin(k_m x), \\ &= \sum_m \chi_m(t)\phi_m(x), \end{aligned}$$

$$\begin{aligned} c(x_p, x_q) &= \langle d(x_p,t)\, d(x_q,t)^\dagger \rangle, \\ &= \sum_m \langle \chi_m \chi_m^\dagger \rangle\, \phi_m(x_p)\phi_m(x_q), \\ &= \hat{c}(\Delta x), \qquad \hat{c}(\Delta x) = \sum_n A_n \cos(k_n \Delta x), \end{aligned}$$

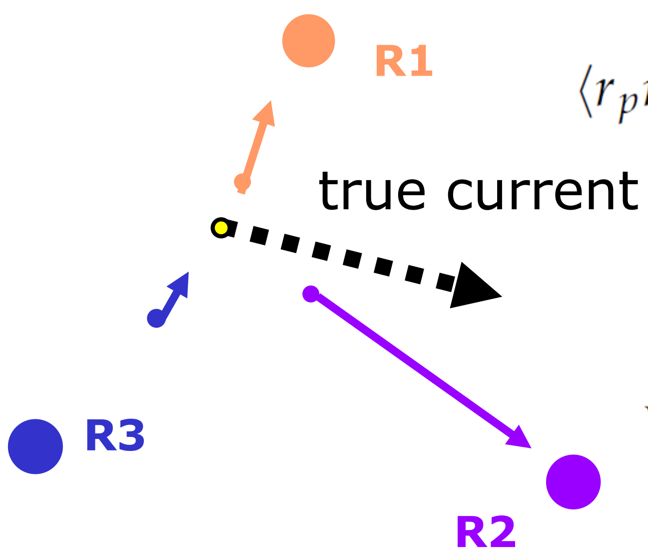
Assumptions: Stationary and homogeneous fields

Covariance estimates (1D, scalar)



$$\int_{-\infty}^{\infty} e^{-\frac{|x|}{\lambda}} e^{-ikx} \mathrm{d}x = \frac{2\lambda}{1+\lambda^2 k^2}$$

Covariance estimate (2D, vector)



$$r(t) = \mathbf{g}^{\dagger}\mathbf{w}(t) + \epsilon(t) = u(t)\cos\theta + v(t)\sin\theta + \epsilon(t),$$

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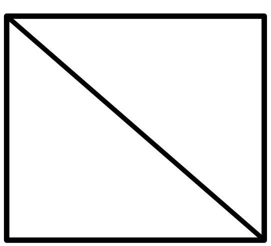
$$\mathbf{d} = \mathbf{G}\mathbf{m} + \mathbf{e}, \tag{14}$$



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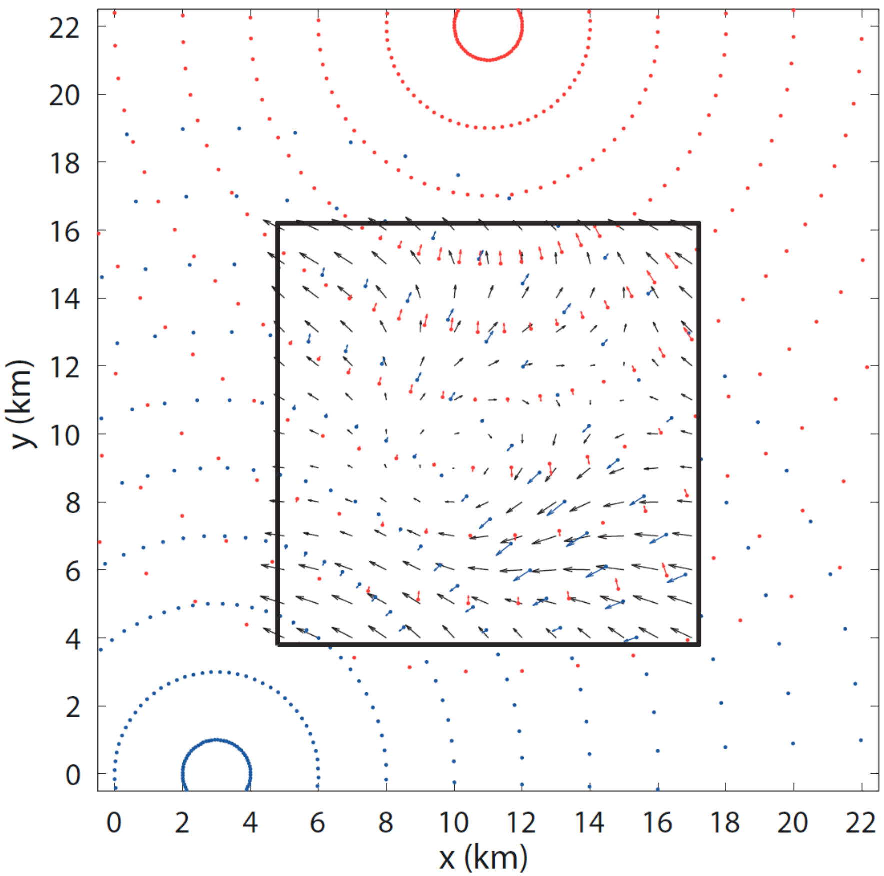
An idealized and spectral model



$$u(x,y,t) = \sum_{m=-M^*}^{M^*} \sum_{n=-N^*}^{N^*} \sum_{s=-S^*}^{S^*} \hat{A}_{mns} \cos \vartheta_{mns} + \hat{B}_{mns} \sin \vartheta_{mns}, \tag{A9}$$

$$v(x,y,t) = \sum_{m=-M^*}^{M^*} \sum_{n=-N^*}^{N^*} \sum_{s=-S^*}^{S^*} \hat{C}_{mns} \cos \vartheta_{mns} + \hat{D}_{mns} \sin \vartheta_{mns}, \tag{A10}$$

where

$$\vartheta_{mns} = k_m x + l_n y - \sigma_s t = 2\pi \left(\frac{m}{L_x} x + \frac{n}{L_y} y \right) - \sigma_s t, \tag{A11}$$

$$\hat{A}_{mns} = (\xi_{mn})^{\frac{1}{2}} \xi_s \mathrm{N}(0,1), \tag{A12}$$

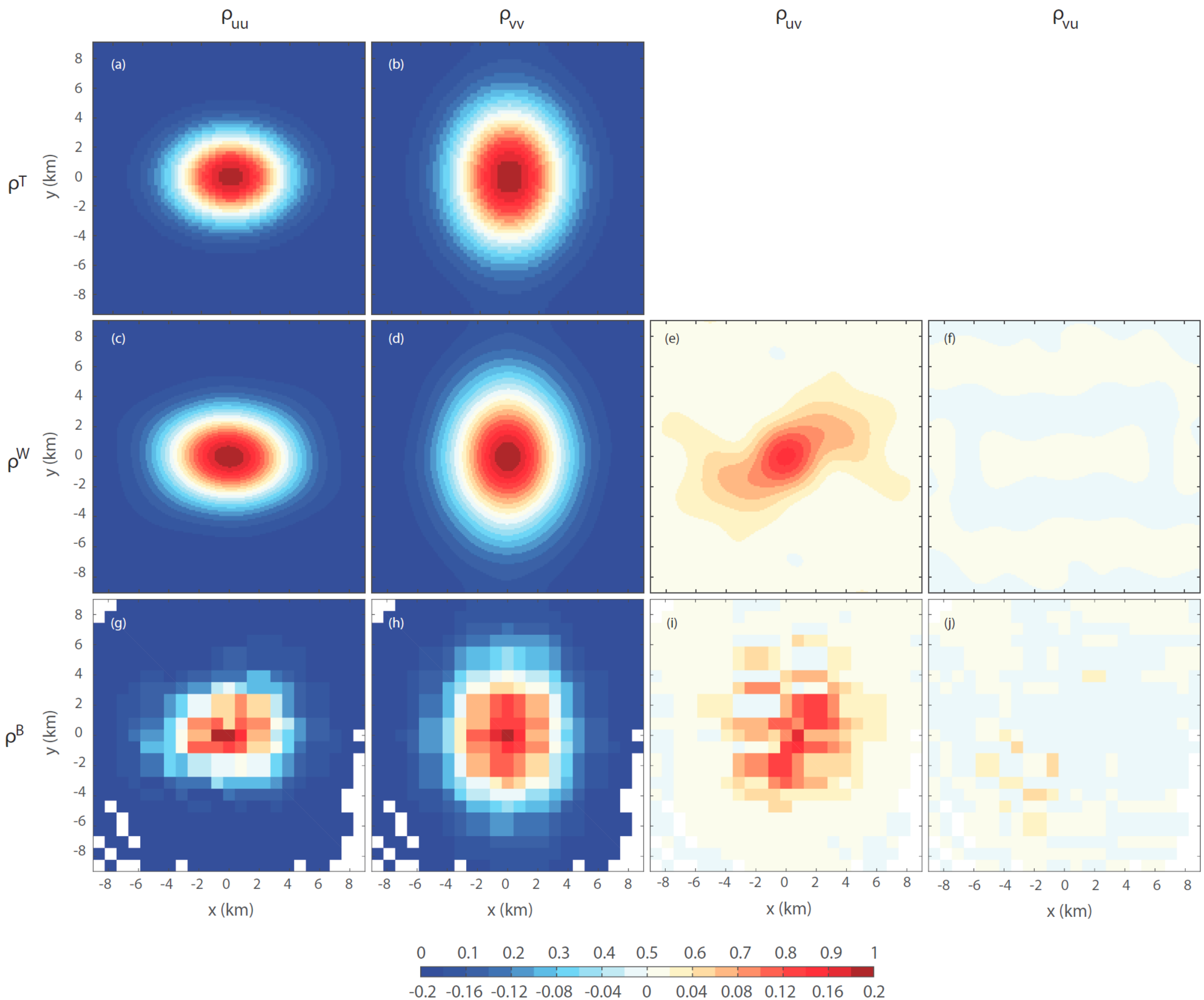
with the power spectrum of the current field in the wavenumber $[\xi_{mn} = \xi(k_m, l_n)]$ and frequency $[\xi_s = \xi(\sigma_s)]$ domains. The wavenumber spectrum can be approximated with a two-dimensional Gaussian or exponential function:

$$\xi(k_m, l_n) = \pi \lambda_x \lambda_y \exp\left(-k_m^2 \lambda_x^2 - l_n^2 \lambda_y^2\right), \tag{A13}$$

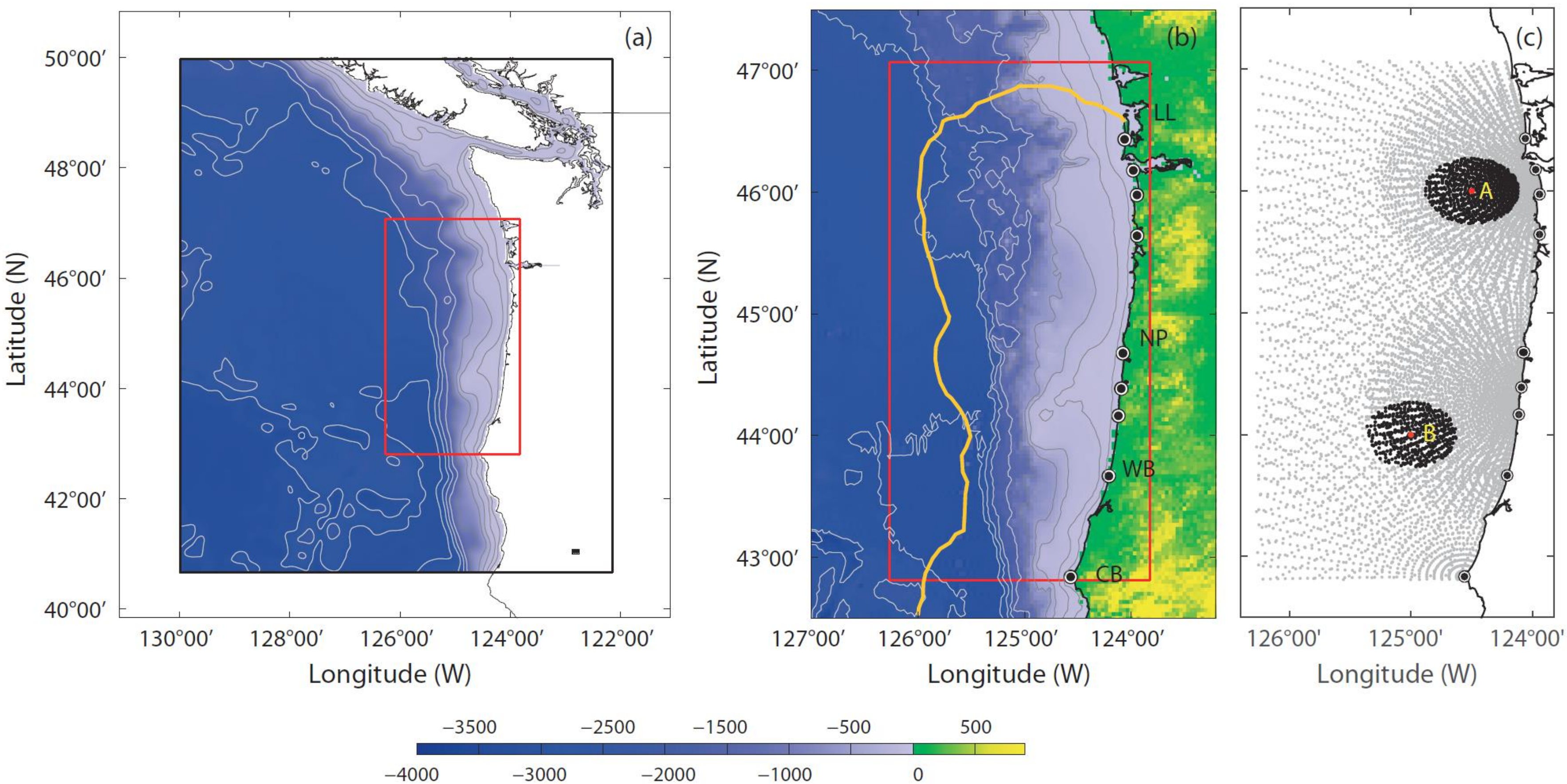
$$\xi(k_m, l_n) = \frac{4\lambda_x \lambda_y}{(1 + k_m^2 \lambda_x^2 + l_n^2 \lambda_y^2)^{3/2}}, \tag{A14}$$

and λ_x and λ_y denote the de-correlation length scales in the x and y directions, respectively.

Covariance estimates (2D, vector)



Numerical model domain



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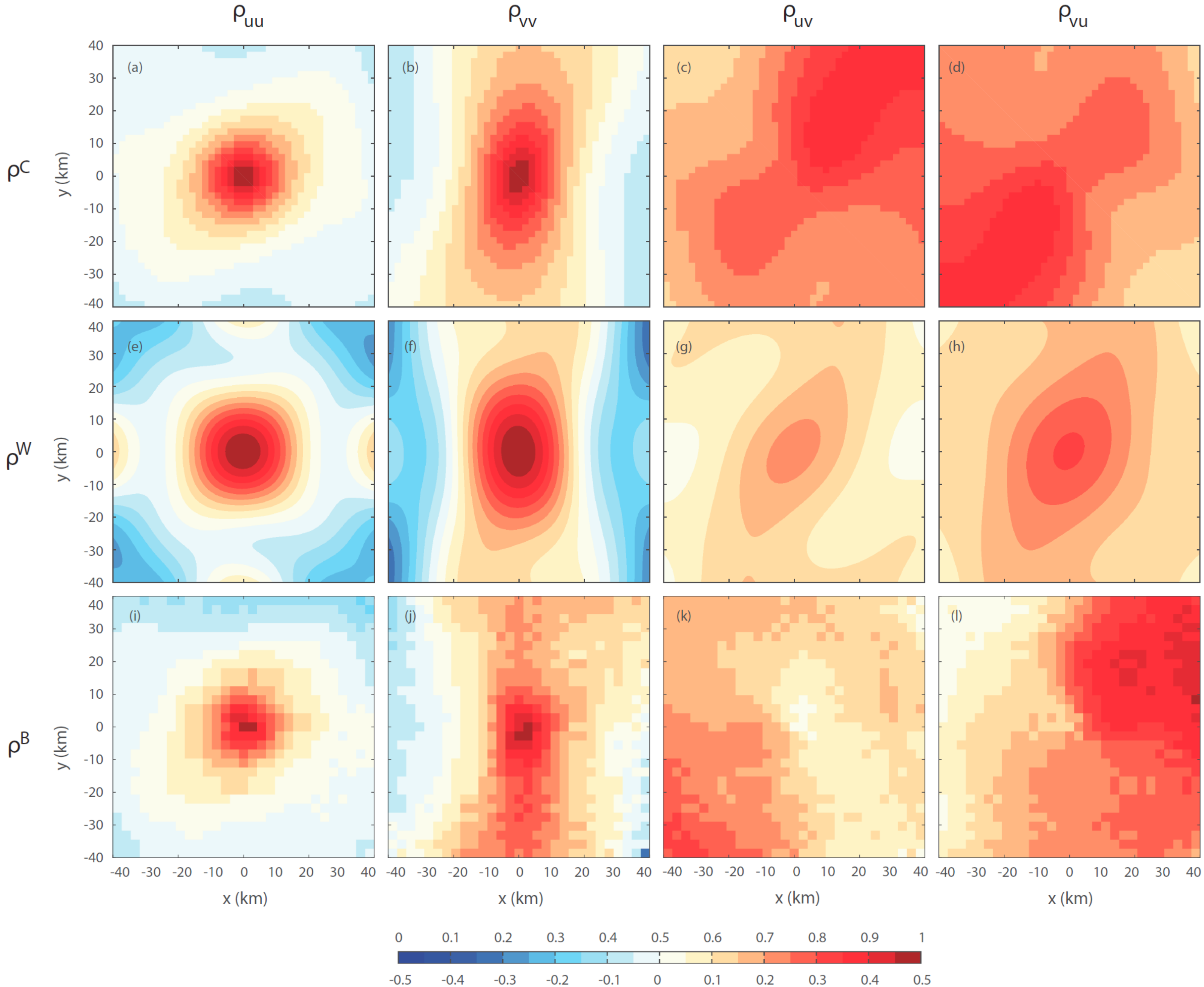
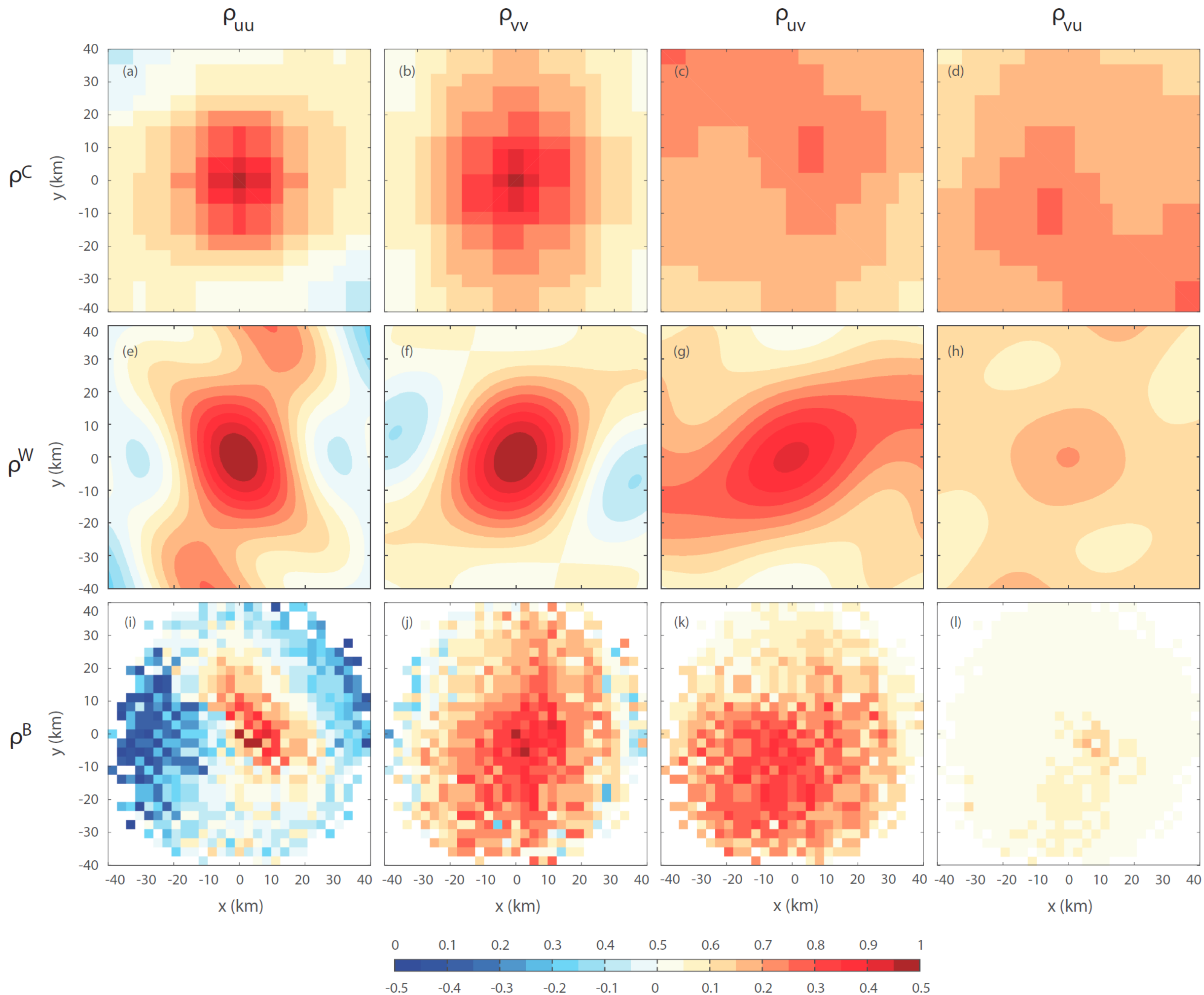
Covariance estimates (2D, vector) - model



Covariance estimates (2D, vector) - observations



Summary

- A direct method of spatial covariance estimates on the scalar and vector data can minimize the bias due to an intermediate step of gridding.
- Solving the inverse problem can be computationally expensive, but we may use advanced computational resources.
- The number of realizations and the density of unevenly samplings (sparse vs. dense) can affect the performance of the proposed analysis, which should be examined with 1D/simple examples.